Advanced Analytic Methods in Applied Mathematics, Science, and Engineering

By Hung Cheng

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Preface

For almost four decades, the Department of Mathematics at MIT has offered a course called "Advanced Analytic Methods in Science and Engineering."[1] The purpose of this course is to strengthen the mathematical background of all entering graduate students, so they may be better prepared for their respective coursework and specialties.

During the past ten years I've been teaching this course I occasionally wrote notes for the students in my class, intended only to supplement the course's textbooks. However, at the end of one recent semester several students suggested I make my materials, which were by that time more than simply "notes," accessible to students beyond MIT. This textbook is the result of expansion of and revisions to that material.

The background of students taking the course is usually fairly diverse. Many of them lack some of the fundamentals that would prepare them for a graduate math course. The first five chapters owe their origin to the need for helping such students, bringing them up to speed. The last five chapters contain more advanced materials.

Teachers and students will thus find that this book's content is flexible enough to meet the needs of a variety of course structures. For a one-semester course with emphasis on approximate methods, a teacher may just skim over the first five chapters, leaving the students to read in more detail the parts they need most. Such a plan would be especially useful for graduate students entering a Ph.D. program in engineering, science, or applied mathematics. But if this book is adopted for a course in advanced calculus for undergraduate engineering, science, or applied mathematics students, then Chapters 1–6 should be emphasized. Chapters 1–5, plus a few selected later chapters, would be suitable for a graduate course for Master's degree students. In addition, Chapters 3, 4, and 5 may be used as part of the materials for a course on partial differential equations.

[1] The course was created by Professor Harvey Greenspan. In 1978 Carl M. Bender and Steven A. Orszag, two lecturers of this course, authored a textbook, *Advanced Mathematical Methods for Scientists and Engineers* (New York, McGraw-Hill, Inc.; reprinted by Springer-Verlag New York, Inc., 1998).

While most graduate students and upper-class undergraduate students have already had a full semester of ordinary differential equations, some of them may need a refresher. Therefore, Chapter 1 includes a very brief summary of ordinary differential equations. This first chapter is also a convenient place to reintroduce the elementary but powerful operator method to students. The operator method enables one to more quickly produce the particular solutions of certain linear ordinary differential equations as well as partial differential equations, and it also facilitates many other calculations.[2]

Chapter 2 is for students who need a quick summary of some of the relevant materials in complex analysis. The important but often neglected subjects of branch points and branch cuts are included, as well as a short discussion of the Fourier integral, the Fourier series, and the Laplace transform.

Many of the analytic methods discussed in this book arose from the need to solve partial differential equations. To help the reader see that connection, Chapters 3, 4, and 5 address partial differential equations.

Because many problems encountered in real life are often not solvable in a closed form, it will benefit a student to learn how to do approximations. Chapter 6 presents the methods of series solutions. A few well-known special functions are used as examples in order to help students gain some familiarity with these functions while learning the methods of series solutions. I will address the topic of irregular singular points of an ordinary differential equation, which is not usually covered in standard textbooks on advanced calculus, such as F. B. Hildebrand, *Advanced Calculus for Applications*, Prentice Hall, 1976. The series solution expanded around an irregular singular point of an integral rank is generally divergent and leads naturally to the concept of asymptotic series, which we'll cover in subsequent chapters.

Chapter 7 discusses the WKB method. This method gives good approximate solutions to many linear ordinary differential equations with a large parameter or those with coefficients that are slowly varying. It is also helpful for yielding solutions near an irregular singular point of a linear differential equation. While the lowest-order WKB solutions are obtained by solving nonlinear differential equations, the higher-order WKB approximations are obtained by iterating linear differential equations. The last section of this chapter discusses the solutions near a turning point.

Chapter 8 addresses the Laplace method, the method of stationary phase, and the saddle point method, which are useful for finding the asymptotic series of

[2]While this method has been routinely used in field theories, particularly with the derivation of various Green functions, it has not been adequately covered in most undergraduate textbooks, with the notable exception of *Differential Equations* by H. T. H. Piaggio, G. Bell and Sons, Ltd., London, 1946 (reprinted in the U.S. by Open Court Publishing Company, LaSalle, Illinois, 1948).

integrals with a large parameter. In the saddle point method, we deviate from the rigorous approach of finding the path of steepest descent. Instead, we advocate finding just a path of descent, as this may somewhat reduce the solution's chores.

Chapters 9 and 10 address the subjects of regular perturbation and singular perturbation. Chapter 9 is devoted to the topic of boundary layers, and Chapter 10 covers the topic of small nonlinear oscillations.

Throughout this book I emphasize a central theme rather than peripheral details. For instance, before discussing how to solve a class of advanced problems, I relate it to the basics and, when possible, make comparisons with similar but more elementary problems. As I demonstrate a method to solve a certain class of problems, I start with a simple example before presenting more difficult examples to challenge the minds of the students. This process gives students a firmer grasp of the subject, enabling them to acquire the key idea more easily. Hopefully, ours will make it possible for them to do mathematics without the need of memorizing a large number of formulae. In the end I hope that they will know how to approach a general problem; this is a skill that leaves students better prepared to treat problems unrelated to the ones given in this book, which they'll likely encounter in their future academic or professional lives.

During my classroom lectures, I emphasize interaction with the students. I often stop lecturing for a few minutes to pose a question and ask everyone to work through it. I believe this method helps to encourage students to learn in a more thorough way and to absorb concepts more effectively, and this book reflects that interactive approach; many "Problems for the Reader" are found throughout the text. To deepen their understanding of the themes that they're learning, students are encouraged to stop and work on these problems *before* looking at the solutions that follow.

This book also passes on to learners some of the problem-solving methods I've developed through the years. In particular, parts of Chapters 9 and 10 offer techniques, which I hope will benefit students and researchers alike. Indeed, I believe that the renormalization methods given in Chapter 10 are more powerful than other methods treating problems of non-linear oscillations so far available.

I am indebted to the group of students who encouraged me to publish this book. Several students have read the field test version of this book and have given me their very helpful suggestions. They include Michael Demkowicz, Jung Hung Lee, Robin Prince, and Mindy Teo. Also, Dr. George Johnston read Chapter 7 and gave me very useful comments. I want to thank Professor T. T. Wu of Harvard University, who introduced me to the saddle point method several decades ago with a depth I had never fathomed as a graduate student. I thank Mr. David Hu for the graphs in Chapters 2 and 8. Special thanks are due to Dr. Dionisios Margetis for graphs in Chapter 9 and the compilation of an extensive bibliography, and to Mr. Nikos Savva for graphs in Chapters 3 and 9. I also am truly grateful to Professor John Strain for his inexhaustible efforts in reading through all of the chapters in the first draft. I am greatly indebted to Dr. H. L. Hu for the many graphs he tirelessly drew for this book.

Chapter 2

Complex Analysis

A. Complex Numbers and Complex Variables

In this chapter, I give a short discussion of complex numbers and the theory of a function of a complex variable.

Before we get to complex numbers, let me first say a few words about real numbers.

All real numbers have meanings in the real world. Ever since the beginning of civilization, people have found great use of real positive integers, say 2 and 30, which came up in conversations such as "my neighbor has two pigs, and I have thirty chickens." The concept of a negative real integer, say -5, is not quite as easy, but it became relevant when a person owed another person five copper coins. It was also natural to extend the concept of integers to rational numbers. For example, when six persons share equally a melon, the number describing the fraction of melon each of them has is not an integer but the rational number $1/6$. When we add, subtract, multiply, or divide integers or rational numbers, the result is always an integer or a rational number.

But the need for other real numbers came up as mathematicians pondered the length of the circumference of, say, a circular city wall. To express this length, the real number π must be introduced. This real number is neither an integer nor a rational number, and is called an irrational number. Another well-known irrational number found by mathematicians is the constant e.

Each of the real numbers, be it positive or negative, rational or irrational, can be geometrically represented by a point on a straight line. The converse is also true: a point on a straight line can always be represented by a real number.

When we add, subtract, multiply, or divide two real numbers, the outcome is always a real number. Thus the root of the linear equation

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$$ax + b = c,$$

with a, b, and c real numbers, is always a real number. That is to say that if we make nothing but linear algebraic operations of real numbers, what comes out is invariably a real number. Thus the real numbers form a closed system under linear algebraic operations.

But as soon as we get to nonlinear operations, the system of real numbers alone becomes inadequate. As we all know, there are no real numbers that satisfy the quadratic equation

$$x^2 = -1.$$

Thus we use our imagination and denote i as a root of this equation. While we have gotten to be comfortable with the imaginary number i, the concept of the imaginary number was not always easy. Indeed, even Gauss once remarked that the "true metaphysics" of i was "hard."

The number

$$\alpha = a + ib,$$

where a and b are real numbers, is called a complex number. The numbers a and b are called the real part and the imaginary part of α, respectively.

While complex numbers might have once appeared to have no direct relevance in the real world, people have since found that the use of complex numbers enables them to handle more easily many physical problems in classical physics. For example, electrical engineers use the imaginary number i extensively, except that they call it j. And at the turn of the twentieth century, complex numbers became almost indispensable with the invention of quantum mechanics.

Let us enter the never-never land of the complex variable z denoted by

$$z = x + iy,$$

where x and y are real variables and

$$i^2 = -1.$$

The complex conjugate of z will be denoted as

$$z^* = x - iy.$$

The variable z can be represented geometrically by the point (x, y) in the Cartesian two-dimensional plane. In complex analysis, this same two-dimensional plane is called the complex plane. The x axis is called the real axis, and the y axis is called the imaginary axis. Let r and θ be the polar coordinates. Then we have

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$$x = r\cos\theta, \quad y = r\sin\theta \; (r \geq 0).$$

The variable θ is defined modulo an integral of 2π. For many functions, the common way is to define θ to be either between 0 and 2π or between $-\pi$ and π. This will be further discussed in Section E of this chapter.

Expressed by the polar coordinates, z is

$$z = r(\cos\theta + i\sin\theta). \tag{2.1}$$

The Euler's formula says

$$e^{i\theta} = \cos\theta + i\sin\theta. \tag{2.2}$$

Thus we have

$$z = re^{i\theta}. \tag{2.3}$$

This is known as the polar form of z. The quantity $r = \sqrt{x^2 + y^2}$ is called the absolute value or the magnitude of z, which is also expressed as $|z|$. The quantity $\theta = \tan^{-1} y/x$ is called the argument or the phase of z.

Incidentally, (2.2) shows that $\cos\theta$ and $\sin\theta$ are respectively the real part and the imaginary part of $e^{i\theta}$, provided that θ is real. Note that the absolute value of $e^{i\theta}$ is

$$\sqrt{\cos^2\theta + \sin^2\theta} = 1.$$

If we set $\theta = 2n\pi$ in (2.2), where n is an integer, we get

$$e^{i2n\pi} = 1.$$

This result can be understood geometrically. The complex number $e^{i2n\pi}$ has the phase $2n\pi$, and is hence located on the positive real axis. This complex number has the magnitude unity, and is hence one unit distance away from the origin. Therefore it is equal to 1.

Setting θ in (2.2) to $(2n + 1)$, where n is an integer, we get

$$e^{i(2n+1)\pi} = -1.$$

Geometrically, $e^{i(2n+1)\pi}$ has the magnitude unity and the phase $(2n+1)\pi$, and is hence located at the negative real axis one unit distance away from the origin. Therefore it is equal to -1.

Setting $n = 1$, we have

$$e^{i\pi} = -1.$$

As we have mentioned, e and π are irrational numbers and there is no simple formula connecting these two numbers. Yet after we introduce the imaginary number i, which is a figment of our imagination, these three numbers are neatly joined together.

Since the argument of z is defined modulo an integral multiple of 2π, the polar form (2.3) can be written as

$$z = re^{i(\theta + 2n\pi)}, \tag{2.4}$$

where n is an integer. Indeed, we have just shown that the factor $e^{i2n\pi}$ in (2.4) is equal to unity and hence (2.4) agrees with (2.3).

Problem for the Reader

Where is the complex number $(1 + i)$ in the complex plane? What are the phase and the magnitude of $(1 + i)$?

Solution

The Cartesian coordinates of the complex number $(1+i)$ are $(x, y) = (1, 1)$. Thus we put a dot on the point $(1, 1)$ in the xy-plane to represent this complex number geometrically.

Inspecting the location of this dot, we find that the phase and the magnitude of $1 + i$ are $\pi/4$ and $\sqrt{2}$, respectively.

The polar form is particularly convenient to use for carrying out the operations of multiplication or division of complex numbers. Let

$$z_1 = r_1 e^{i\theta_1},\ z_2 = r_2 e^{i\theta_2},$$

then

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)},\ z_1/z_2 = (r_1/r_2) e^{i(\theta_1 - \theta_2)}.$$

The formulae above say that the absolute value of the product (ratio) of two complex numbers is equal to the product (ratio) of the absolute values of these

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complex numbers, and the phase of the product (ratio) of two complex numbers is equal to the sum (difference) of the phases of these complex numbers. These operations would have been a little more cumbersome to carry out if we had expressed the complex numbers with the Cartesian form.

Needless to say, using the polar form to do multiplication and division of more factors of complex numbers is even more laborsaving. In particular, we have

$$z^m = r^m e^{im\theta} = r^m(\cos m\theta + i \sin m\theta).$$

Setting $r = 1$, we obtain from the formula above that

$$(\cos\theta + i \sin\theta)^m = \cos m\theta + i \sin m\theta.$$

This single identity contains a number of identities we are familiar with. For example, choosing m to be 2, we get from this identity

$$\cos^2\theta - \sin^2\theta + 2i \cos\theta \sin\theta = \cos 2\theta + i \sin 2\theta.$$

By equating the real parts as well as the imaginary parts of the two sides of the equation above, we get the familiar identities

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

and

$$\sin 2\theta = 2\cos\theta \sin\theta,$$

which express $\cos 2\theta$ and $\sin 2\theta$ as quadratic forms of $\cos\theta$ and $\sin\theta$.

Problem for the Reader

Find the roots of the equation

$$e^{e^z} = 1.$$

Solution

Let us call

$$\omega \equiv e^z;$$

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then the equation under consideration is

$$e^{\omega} = 1.$$

This problem is easily solved if we write the equation as

$$e^{\omega} = e^{2n\pi i},$$

where n is any integer, and equate the exponents of the two sides of the equation. We get

$$\omega = 2n\pi i,$$

or

$$e^{z} = 2n\pi i,$$

where n is any integer. We shall solve the equation above for z. If $n = 0$, the equation above is

$$e^{z} = 0.$$

Since $|e^{z}| = e^{x}$, which is never equal to zero unless x is equal to minus infinity, $e^{z} = 0$ has no finite root.

If $n > 0$, the equation above is

$$e^{z} = e^{\ln(2n\pi)} e^{i\pi/2} e^{2m\pi i}.$$

Equating the exponents of the two sides, we get

$$z = \ln(2n\pi) + i\pi(1/2 + 2m),$$

where m is any integer.

Similarly, if $n < 0$, we get

$$z = \ln(2\,|n|\,\pi) + i\pi(-1/2 + 2m).$$

There are a doubly infinite number of solutions for equation $e^{e^{z}} = 1$.

Problem for the Reader

Find the phase and the magnitude of the power function z^{a}, where a is a number that is not an integer.

Solution

We again use the polar form (2.4) for z and get

$$z^a = \left[re^{i(\theta+2n\pi)}\right]^a = r^a e^{i(\theta+2n\pi)a},\ (n = 0, \pm1, \pm2, \cdots).$$

The identity above shows that the magnitude of z^a is r^a, and the phase of z^a is equal to $(\theta + 2n\pi)a$, which has infinitely many values. While n is an integer, na is not necessarily an integer. Thus $e^{i2n\pi a}$ is not necessarily equal to unity and z^a generally has infinitely many values.

Exceptions occur when a is a rational number. Consider for example $z^{1/2}$. Setting $a = 1/2$ and $z = 1$ in the expression above, we get

$$\begin{aligned} 1^{1/2} = e^{in\pi} &= 1,\ n \text{ even}, \\ &= -1, n \text{ odd}, \end{aligned}$$

which is the familiar result that the equation $z^2 = 1$, or $z = 1^{1/2}$, has two roots: 1 and -1.

Similarly, setting $a = 1/N$ and $z = 1$, we find that $1^{1/N}$ is equal to

$$e^{2n\pi i/N}, n = 0, 1 \cdots (N-1),$$

with the other integral values of n giving no new roots. This corresponds to the familiar result that the equation

$$\omega^N = 1$$

has N roots.

B. Analytic Functions

Consider the limit

$$\lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z} \tag{2.5}$$

for a complex-value function $f(z)$, where

$$\Delta f = f(z + \Delta z) - f(z).$$

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The limit of the ratio above, if it exists, is called the derivative of $f(z)$, and is denoted as $f'(z)$.

The function $f(z)$ is said to be analytic at z_0 if $f'(z)$ exists in a neighborhood of z_0.The function $f(z)$ is said to be analytic in a region R in the complex z-plane if $f'(z)$ exists for every point z in R.

While (2.5) resembles the definition of the derivative of a function of a real variable x

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x},$$

there is a substantive difference between them. The point is that Δz has both a real part and an imaginary part, i.e., $\Delta z = \Delta x + i\Delta y$. Therefore, if $f(z)$ is to have a derivative, the limit of (2.5) is required to exist for any Δx and Δy, as long as both of them go to zero. There is no restriction, for example, on the ratio of $\Delta y/\Delta x$, which may take any value. This is a strong condition on the function $f(z)$.

A strong condition has strong consequences. Let

$$f(z) = u(x, y) + iv(x, y),$$

where u and v are the real part and the imaginary part of $f(z)$, respectively. Then the expression in (2.5) is

$$\lim_{\Delta z \to 0} \frac{\Delta u + i\Delta v}{\Delta x + i\Delta y}, \tag{2.6}$$

where

$$\Delta u = u(x + \Delta x, y + \Delta y) - u(x, y),$$

and similarly for Δv.

We first consider the limit of (2.6) with Δz real, i.e., $\Delta z = \Delta x$. Then the limit of (2.6) is equal to

$$\lim_{\Delta x \to 0} \frac{\Delta u + i\Delta v}{\Delta x}\bigg|_{y \text{ fixed}} = u_x + iv_x, \tag{2.7}$$

where u_x, for example, is the partial derivative of u with respect to x. Next we consider the limit (2.6) with Δz purely imaginary, i.e., $\Delta z = i\Delta y$. We have

$$\lim_{\Delta y \to 0} \frac{\Delta u + i\Delta v}{i\Delta y}\bigg|_{x \text{ fixed}} = \frac{(u_y + iv_y)}{i}. \tag{2.8}$$

If $f(z)$ has a derivative, the expressions of (2.7) and (2.8) are, by definition, the same. This requires that

$$u_x = v_y,\ u_y = -v_x. \tag{2.9}$$

The equations in (2.9) are known as the Cauchy-Riemann equations. The real part and the imaginary part of an analytic function must satisfy these equations.

While we have only required that the limit of (2.5) is the same with Δz either purely real or purely imaginary, it is straightforward to prove that this limit is the same for any complex Δz when the Cauchy-Riemann equations are obeyed. (See homework problem 1 of this chapter.)

Differentiating the first Cauchy-Riemann equation with respect to x, we get

$$u_{xx} = v_{yx}.$$

Differentiating the second Cauchy-Riemann equation with respect to y, we get

$$u_{yy} = -v_{xy}.$$

Adding these two equations together, we get

$$u_{xx} + u_{yy} = 0.$$

The equation above is called the Laplace equation. We have shown that the real part of an analytic function must satisfy the Laplace equation.

We may similarly prove that v, the imaginary part of an analytic function, also satisfies the Laplace equation.

The Laplace equation is an important equation in physical sciences. From what we have just discussed, one may find the solution of a two-dimensional Laplace equation satisfying certain boundary conditions by looking for the analytic function the real part (or the imaginary part) of which satisfies these boundary conditions.

A function satisfying the Laplace equation is said to be harmonic. Thus the real part and the imaginary part of an analytic function are always harmonic. We call u and v the harmonic conjugate of each other.

Problem for the Reader

Find the real part and the imaginary part of e^z and show that they satisfy the Cauchy-Riemann equations.

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Solution

We have

$$e^z = e^{x+iy} = e^x e^{iy} = u(x,y) + iv(x,y).$$

Using Euler's formula for e^{iy}, we get

$$u(x,y) = e^x \cos y,\ v(x,y) = e^x \sin y.$$

Thus

$$u_x = e^x \cos y,\ u_y = -e^x \sin y,$$
$$v_x = e^x \sin y,\ v_y = e^x \cos y.$$

We see that u and v satisfy the Cauchy-Riemann equations for all values of z. Thus the function e^z is analytic for all values of z. Incidentally, a function that is analytic at all points in the finite complex plane is called an entire function of z.

Problem for the Reader

Is the function $f(z) = zz^*$ analytic?

Solution

For the function zz^*,

$$u(x,y) = x^2 + y^2,\ v(x,y) = 0.$$

We have

$$u_x = 2x,\ u_y = 2y,\ v_x = v_y = 0.$$

Thus the Cauchy-Riemann equations are not satisfied except at the origin, which is but a point, not a region. Since the derivative of the function exists for no neighborhood of the origin, it is not analytic, even at the origin.

An intuitive way to understand why zz^* is not analytic is to think of this function as dependent on z^*.We shall show that an analytic function is independent of z^*.

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To see this, let us first mention that we usually think of z^* to be dependent on z, and a function of z and z^* is just a function of z. Indeed,

$$\Delta z = \Delta x + i\Delta y,\ \Delta z^* = \Delta x - i\Delta y.$$

Since the magnitude of Δz is equal to that of Δz^*, we conclude that if Δz is nonzero, Δz^* is nonzero, which makes it impossible to vary z when keeping z^* fixed.

But this is true because we have implicitly accepted the premise that both Δx and Δy are real. As a matter of fact, that $\Delta z^* = 0$ implies

$$\Delta x = i\Delta y.$$

Therefore, if Δx and Δy are allowed to be complex, it is possible to have Δz be nonzero with Δz^* equal to zero. For example, when Δy is real, Δz^* is zero if Δx is equal to the imaginary number $i\Delta y$. With this provision we may regard z and z^* as independent variables.

We have

$$x = \frac{z + z^*}{2},\ y = \frac{z - z^*}{2i}.$$

A function of the two variables x and y can now be considered as a function of z and z^*. The partial differentiation with respect to z can be defined with the chain rule of partial differentiation. We have

$$\frac{\partial}{\partial z} = \frac{\partial x}{\partial z}\frac{\partial}{\partial x} + \frac{\partial y}{\partial z}\frac{\partial}{\partial y} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right).$$

Similarly,

$$\frac{\partial}{\partial z^*} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

Let us now get back to the point that prompted this discussion. Let f be a complex-valued function of x and y. Let the real part and the imaginary part of f be denoted as u and v, respectively. We have

$$\frac{\partial}{\partial z^*}f = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(u + iv) = \frac{u_x - v_y + i(u_y + v_x)}{2}.$$

If f is an analytic function, then u and v satisfy the Cauchy-Riemann equations and hence

$$\frac{\partial}{\partial z^*}f = 0.$$

This says that an analytic function is independent of z^*.

Problem for the Reader

Show that

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial}{\partial z}\frac{\partial}{\partial z^*}.$$

Solution

$$4\frac{\partial}{\partial z}\frac{\partial}{\partial z^*} = \left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right).$$

It follows that the Laplace equation can be written as $\frac{\partial^z}{\partial z \partial z^*}u = 0$, and the solution of the Laplace equation is the sum of a function of z and a function of z^*.

Next, we give a few examples of functions that are analytic. The power function z^n with n an integer is analytic. While this result may very well be expected, I will give it a proof below. We have, by expressing $(z + \Delta z)^n$ with the binomial expansion,

$$\lim_{\Delta z \to 0} \frac{(z + \Delta z)^n - z^n}{\Delta z} = \lim_{\Delta z \to 0} \frac{nz^{n-1}\Delta z + \cdots}{\Delta z},$$

where the terms unexhibited are at least as small as the square of Δz. The limit above exists for all Δz, and we get

$$\frac{d}{dz}z^n = nz^{n-1},$$

which is the same formula we learned in calculus. Thus the derivative of the power function z^n exists for all values of z. Therefore, z^n is analytic for all values of z, or is an entire function of z.

Since the power function z^n is analytic, so is the linear superposition of a finite number of power functions. Indeed, so is the sum of an infinite number of power functions, as long as this sum is absolutely convergent. Conversely, we shall show in Section C that a function analytic at a point z_0 always has a convergent Taylor series expansion around z_0.

We have learned in calculus that the Maclaurin series for $\sin x$ is

$$\sin x = x - x^3/(3!) + x^5/(5!) + \cdots .$$

Let us replace x in the series above by z and define $\sin z$ to be this series, i.e.,

$$\sin z \equiv z - z^3/(3!) + z^5/(5!) + \cdots . \tag{2.10}$$

Similarly, we define

$$\cos z = 1 - z^2/2! + z^4/4! + \cdots . \tag{2.11}$$

We shall use the ratio test to prove that the series in (2.11) is absolutely convergent for all values of z. Let the n^{th} term in the series of (2.11) be a_n. Then the ratio

$$\frac{a_n}{a_{n-1}} = -\frac{z^2}{(2n-2)(2n-3)}$$

vanishes in the limit $n \to \infty$ for all values of z. Since the ratio of the n^{th} term and the $(n-1)^{\text{th}}$ term vanishes as $n \to \infty$ for all z, the ratio test asserts that the series of (2.11) converges absolutely for all values of z. Therefore, $\cos z$ defined by (2.11) is meaningful for all z. Being the same as $\cos x$ when $z = x$, $\cos z$ is called the analytic continuation of $\cos x$ into the complex plane. The analytic continuation of a function from the real line into the complex plane is unique. (For more general considerations of analytic continuation of a function, see homework problem 9.) In the case of $\cos z$, this means that the function defined in (2.11) is the only function possible that is analytic everywhere and agrees with $\cos x$ when $z = x$.

Similarly, the series in (2.10) is absolutely convergent for all z. Therefore, $\sin z$ defined by (2.10) is the unique analytic continuation of $\sin x$ into the complex plane.

C. The Cauchy Integral Theorem

The contour integral

$$I = \int_c f(z)dz,$$

where c is a contour in the complex plane, is defined to be

$$\int_c (u+iv)(dx+idy) = \int_c (udx - vdy) + i\int_c (udy + vdx). \tag{2.12}$$

We note that the two integrals on the right side of (2.12) are line integrals in the two-dimensional plane, which we already encountered in calculus.

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An example of a line integral is the work done by a force. As we know, the work done by moving a particle from (x, y) to $(x + \Delta x, y + \Delta y)$ against a force

$$\overrightarrow{F} = M(x, y)\overrightarrow{i} + N(x, y)\overrightarrow{j}$$

is equal to the scalar product of the force with the displacement vector

$$\Delta x\overrightarrow{i} + \Delta y\overrightarrow{j}.$$

Thus the work done is

$$M(x, y)\Delta x + N(x, y)\Delta y.$$

Therefore, if A and B are two points in the xy-plane, the work done in moving a particle from A to B along a path c against the force is equal to the line integral

$$\int_c (Mdx + Ndy).$$

We also recall that $\overrightarrow{F}$ is known as a conservative force if the curl of $\overrightarrow{F}$ vanishes. Equivalently, $\overrightarrow{F}$ is a conservative force if there exists a potential V such that

$$\overrightarrow{F} = -\overrightarrow{\nabla} V.$$

If $\overrightarrow{F}$ is conservative, the work done in moving a particle from one point to another depends only on the difference of the values of the potential at these two points, and is independent of the path. To say this more precisely, let the potential V exist in a region R in the two-dimensional plane; then

$$\int_{c_1} (Mdx + Ndy) = \int_{c_2} (Mdx + Ndy),$$

provided that c_1 and c_2 are two curves with the same endpoints and both lie inside R.

If the potential V exists, we have

$$M = -V_x, \ N = -V_y,$$

and hence

$$M_y = N_x. \tag{2.13}$$

The converse is indeed also true: if (2.13) holds in a region R, then the force is the gradient of a potential.

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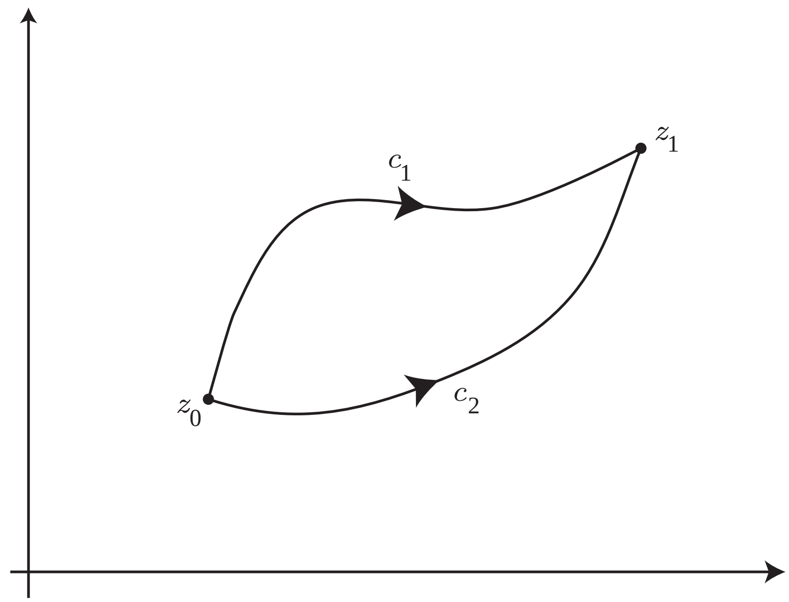

Figure 2.1

Now for the first line integral in (2.12), M is u and N is $-v$. Thus the condition (2.13) for this line integral is the second Cauchy-Riemann equation. For the second line integral in (2.12), M is v and N is u . Thus the condition (2.13) for this line integral is the first Cauchy-Riemann equation. The contour integral I in (2.12) is therefore path independent if $f(z)$ is analytic. More precisely, let c_1 and c_2 be two curves, both join the lower endpoint z_0 to the upper endpoint z_1 in the complex z-plane, and both lie inside the region R where $f(z)$ is analytic. Then we have

$$\int_{c_1} f(z)dz = \int_{c_2} f(z)dz. \tag{2.14}$$

Equation (2.14) tells us that we may deform the contour c_1 to the contour c_2, where c_1 and c_2 have the same endpoints, provided that $f(z)$ is analytic in the region lying between c_1 and c_2.

The contours c_1 and c_2 in (2.14) are open contours. We shall extend (2.14) to closed contours. Let c and c' be closed contours of the same sense of direction, i.e., either both counterclockwise or both clockwise, with no singularities of $f(z)$ lying between c and c'. We choose a point z_0 on c and think of the closed contour c as a contour joining the point z_0 to itself. Let us draw a line joining z_0 to a point z_0' on c', forming a bridge between c and c'. Then we may think of c' as another contour joining z_0 to itself. This is because c' is the contour that begins at

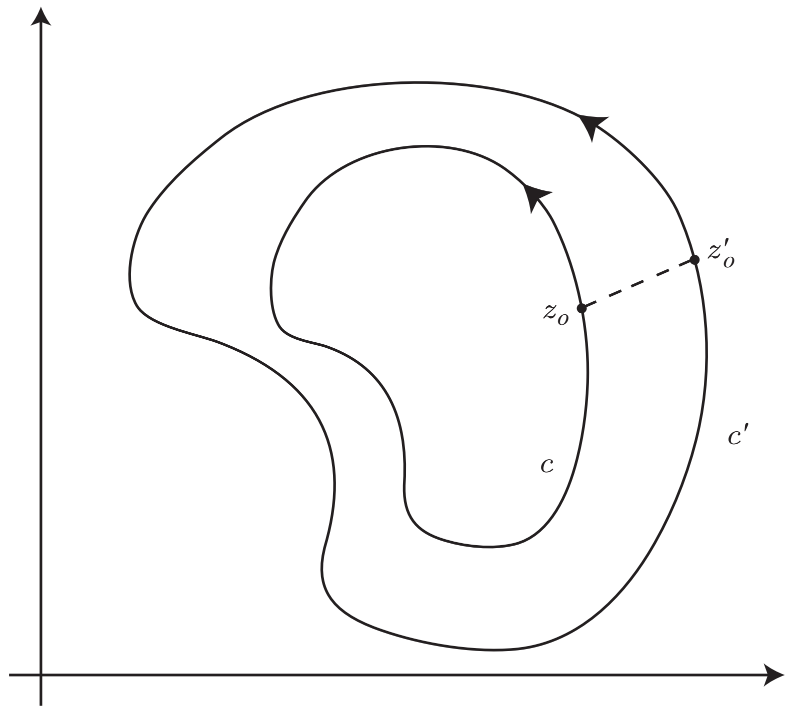

Figure 2.2

z_0, crosses the bridge to z'_0, and follows c' to return to z'_0, then crosses the bridge in the reverse direction to finally come back to z_0. As the bridge is crossed twice in opposite directions, the two contour integrals associated with the contour of the bridge cancel each other. Therefore, c' can also be considered as a closed contour joining z_0 to itself, and by (2.14) we have

$$\oint_c f(z)dz = \oint_{c'} f(z)dz, \tag{2.15}$$

where the symbol $\oint$ denotes an integration over a closed contour.

Equation (2.15) says that the contour c can be deformed into c' provided that $f(z)$ is analytic in the region lying between c and c'.

Let us go from z_0 to z_1 along contour c_1 in Figure 2.1, then go from z_1 back to z_0 along $-c_2$, which is c_2 in the reverse direction. The contour $c = c_1 - c_2$ is a closed contour. Thus (2.14) can be written as

$$\oint_c f(z)dz = 0 \tag{2.16}$$

provided that $f(z)$ is analytic in a region R and c is a closed contour c inside R. Equation (2.16) is the Cauchy integral theorem.

Next, we consider the integral

$$I_n = \oint_c \frac{dz}{(z-z_0)^n},$$

where c is a closed contour in the counterclockwise direction and n is a positive integer. The integrand blows up at $z = z_0$. We say that the integrand has a singularity at z_0. More generally, if a single-value function $f(z)$ is not analytic at point z_0, then we say that $f(z)$ has a singularity at z_0.

If c does not enclose z_0, I_n vanishes by Cauchy's integral theorem. But if c encloses z_0, as is illustrated in Figure 2.3, we may deform the contour into the circle C_R without crossing any singularity of the integrand, where C_R is the circle, the center of which is z_0, and the radius of which is R.

Now a point z on C_R satisfies

$$|z - z_0| = R,$$

and hence the polar form of $z - z_0$ is

$$z - z_0 = e^{i\theta} R.$$

Thus we get

$$dz = ie^{i\theta} R d\theta.$$

Therefore, we have

$$I_n = \frac{i}{R^{n-1}} \int_0^{2\pi} e^{i(1-n)\theta} d\theta.$$

The integral above is easily calculated. Indeed, we have

$$\begin{aligned} \int_0^{2\pi} e^{i(1-n)\theta} d\theta &= 2\pi, \ n = 1, \\ &= 0, \ n \neq 1. \end{aligned}$$

Thus we conclude that, if z_0 is inside the closed counterclockwise contour c, we have

$$\begin{aligned} I_n &= 2\pi i, \ n = 1, \\ &= 0, \ n \neq 1. \end{aligned} \tag{2.17}$$

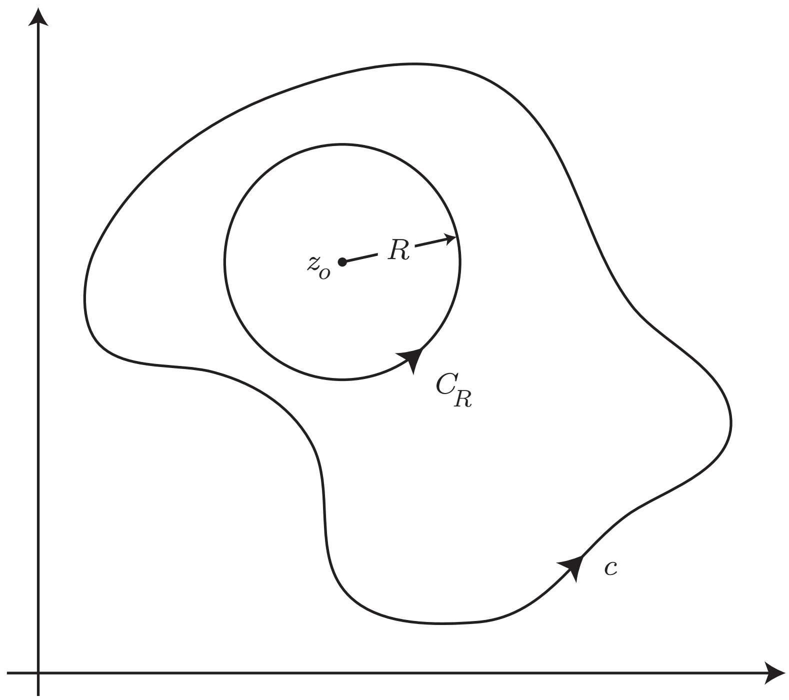

Figure 2.3

Equation (2.17) enables us to derive the Cauchy integral formula, which says that if $f(z)$ is analytic in the region R, and if z is an interior point of R, then we have

$$f(z) = \frac{1}{2\pi i} \oint_c \frac{f(z')}{z' - z} dz', \tag{2.18}$$

provided that c is a closed curve enclosing z once in the counterclockwise direction and lying inside R. To prove this, we deform the contour c into the circle c_ϵ with center at z and radius ϵ. This is allowed, as the integrand of (2.18) is analytic in the region lying between c and c_ϵ. As we make ϵ approach zero, z' approaches z and the integral approaches

$$\frac{f(z)}{2\pi i} \oint_{c_\epsilon} \frac{1}{z' - z} dz'.$$

By (2.17), Cauchy's integral formula is proved.

Differentiating the Cauchy integral formula with respect to z, we obtain the derivative of f as

$$f'(z) = \frac{1}{2\pi i} \oint_c \frac{f(z')}{(z' - z)^2} dz'.$$

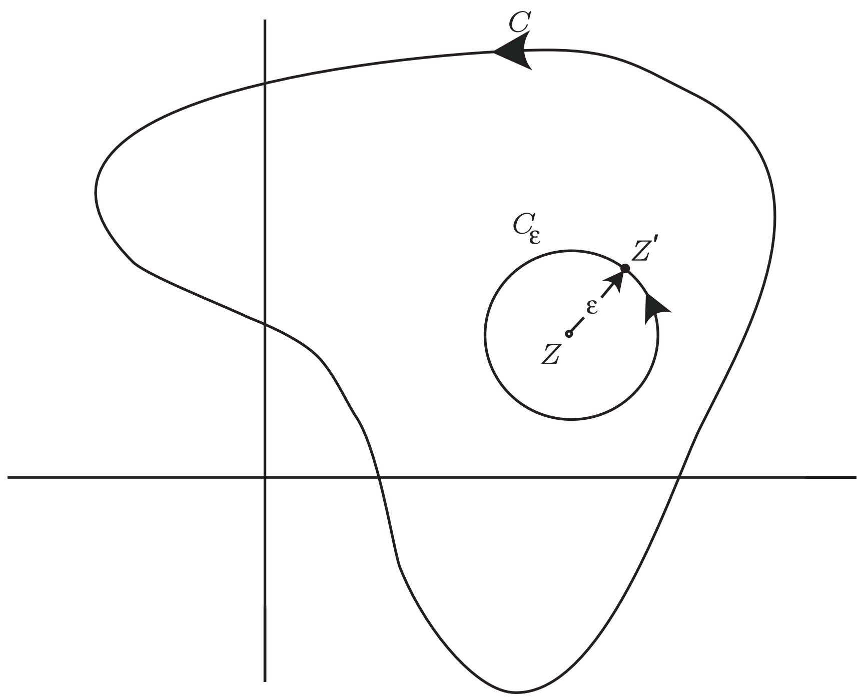

Figure 2.4

By differentiating the Cauchy integral formula n more times, we get

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_c \frac{f(z')}{(z'-z)^{n+1}} dz'.$$

Thus if a function of a complex variable z is analytic, it has derivatives of all orders. But by definition, a function of a complex variable is analytic if it has the first derivative; thus a function of a complex variable has derivatives to all orders if it has the derivative of the first order.

This may seem surprising to readers who have learned in calculus that if a function of a real variable x has a first derivative, it does not necessarily have a second derivative, not to mention even higher-order derivatives. The seeming contradiction is resolved by the fact that the existence of the derivative of a function of a complex variable requires a stronger condition than that of a function of a real variable.

We may now prove that a function $f(z)$ analytic at z_0 has the Taylor series expansion

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$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n, \tag{2.19}$$

where z lies in a neighborhood of z_0 to be specified later. To prove (2.19), we choose c in (2.18) to be C_R, the circle with center at z_0 and radius R, where R is sufficiently large so that c encloses z. Now if z' is a point on C_R,

$$|z' - z_0| = R.$$

Also, since z is inside C_R,

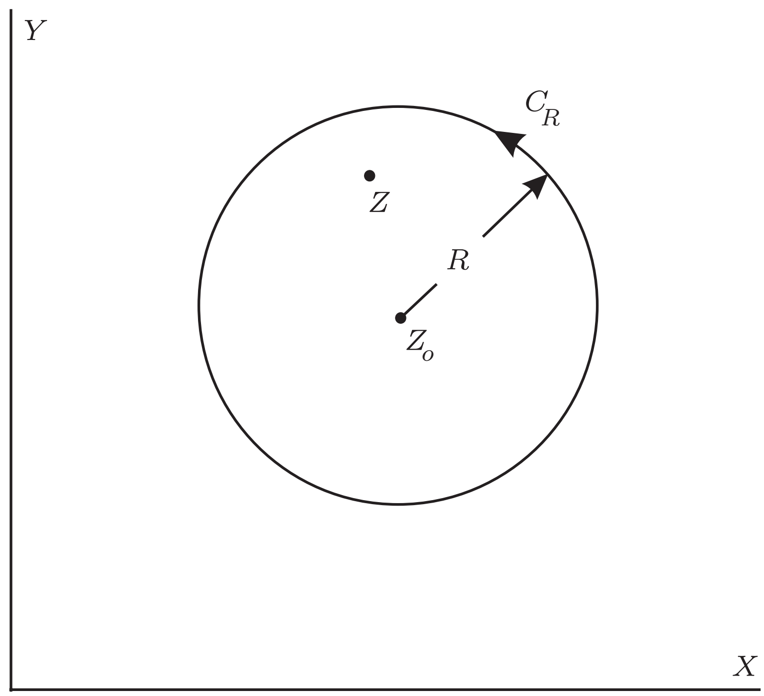

Figure 2.5

$$|z - z_0| < R.$$

Thus we have

$$|z - z_0|/|z' - z_0| < 1.$$

Now

$$(z'-z)^{-1} = \left[(z'-z_0)-(z-z_0)\right]^{-1} = (z'-z_0)^{-1}\left(1-\frac{z-z_0}{z'-z_0}\right)^{-1}.$$

As we know, $(1-\omega)^{-1}$ is equal to the convergent series

$$\sum_{n=0}^{\infty}\omega^n$$

provided that $|\omega|<1$. Identifying ω with $(z-z_0)/(z'-z_0)$, we have,

$$(z'-z)^{-1} = (z'-z_0)^{-1}\sum_{n=0}^{\infty}\left(\frac{z-z_0}{z'-z_0}\right)^n.$$

Substituting the expression above into (2.18), we obtain the Taylor series expansion (2.19).

The contour C_R is a circle inside which $f(z)$ is analytic. Thus C_R is not allowed to enclose any singularity of $f(z)$. Let z_1 be the singularity of $f(z)$ closest to z_0; then the largest value of R possible is $|z_0-z_1|$. Therefore, the radius of convergence of the Taylor series (2.19) is $|z_0-z_1|$.

As an example, since $\sin n\pi$ vanishes for any integral value n, the function $z(\sin\pi z)^{-1}$ has singularities at $z=n$, where n is any integer not equal to zero. The point $z=0$ is not a singularity of $z(\sin\pi z)^{-1}$ because the numerator of this function vanishes at $z=0$. Indeed, by l'Hopital's rule, $z(\sin\pi z)^{-1}$ is equal to π^{-1} at $z=0$. Let us consider the Maclaurin series for $z(\sin\pi z)^{-1}$. Since the singularities of $z(\sin\pi z)^{-1}$ closest to the origin are $z=\pm 1$, this series converges in a circle with center at the origin and radius unity.

As another example, we have shown that the function e^z is an entire function. Therefore, we know that its Maclaurin series

$$e^z = 1+\frac{z}{1!}+\frac{z^2}{2!}+\cdots \tag{2.20}$$

converges for all finite values of z. This may also be directly verified with the use of the ratio test.

Problem for the Reader

Show that

$$\sin(iz) = i\sinh z,\ \cos(iz) = \cosh z.$$

Solution

From the Maclaurin series (2.10) for $\sin z$ we get

$$\sin(iz) = i\left[z + z^3/(3!) + z^5/(5!) + \cdots\right] = i\sinh z.$$

In a similar way, we get

$$\cos(iz) = 1 + z^2/2! + z^4/4! + \cdots = \cosh z.$$

From (2.10), (2.11), and (2.20), we find

$$e^{iz} = \cos z + i\sin z, \tag{2.21}$$

which is Euler's formula for a complex argument. Replacing z in (2.21) by $-z$, we have

$$e^{-iz} = \cos z - i\sin z.$$

Therefore,

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \tag{2.22}$$

and

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}. \tag{2.23}$$

When $z = x$, (2.21), (2.22), and (2.23) are identities already established in calculus. We now see that they also hold when z is complex.

As we have mentioned, an equality between analytic functions valid for real $z = x$ is invariably valid when z is complex. This is a result of the uniqueness of analytic continuation. But a nonanalytic relation that holds when $z = x$ often does not hold when z is complex. For example, the inequality

$$-1 \leq \sin x \leq 1$$

does not hold when x is replaced by the complex variable z. Indeed, if z is complex, $\sin z$ is not even a real number. Note also that $\sin z$ is not the imaginary part of e^{iz} when z is complex, nor is $\cos z$ the real part of e^{iz} when z is complex.

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If $f(z)$ is not analytic at z_0, it has no derivative at z_0 and naturally has no Taylor series expansion around z_0. But it may have another kind of series expansion around z_0. As an example, consider the function $f(z) = e^{1/z}$, which is not analytic at $z = 0$. Nevertheless, we have by (2.20) that

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \cdots . \tag{2.24}$$

The series above is not a Taylor series, as it is not a sum of positive power functions. Instead, it is an example of a Laurent series defined by ((2.25)) below. Since the series for e^z converges for all finite z, the Laurent series for $e^{1/z}$ converges for all $z \neq 0$.

More generally, an analytic function has a Laurent series expansion around an isolated singularity, which we shall define as follows. Let z_0 be a singularity of $f(z)$, and z_1 be the singularity of $f(z)$ closest to z_0. If $|z_0 - z_1|$ is not equal to zero, then z_0 is called an isolated singularity of $f(z)$. Not all singularities of an analytic function are isolated. For example, the singularites of the function $1/\sin(\pi/z)$ are located at $z = 1/n$, $n = 0, \pm 1, \pm 2, \cdots$, where $\sin(\pi/z)$ vanishes. Since $1/n$ for n arbitrarily large is arbitrarily close to the origin, the point $z = 0$ is not an isolated singularity of $1/\sin(\pi/z)$.

If z_0 is an isolated singularity of $f(z)$, we may prove with the use of the Cauchy integral formula that $f(z)$ has the Laurent series expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n, \tag{2.25}$$

which contains not only positive powers of $(z - z_0)$, but also negative powers of $(z - z_0)$.

The series (2.25) is convergent at every point, with the exception of the point z_0, inside the circle with the center at z_0 and with the radius $|z_0 - z_1|$, where z_1 is the singularity of $f(z)$ closest to z_0. This can again be proven with the use of the Cauchy integral formula (2.18). (See homework problem 12.)

We note that the difference between a Taylor series and a Laurent series is that the latter has negative power functions $(z - z_0)^{-m}$, where $m > 0$. These power functions blow up at $z = z_0$. Indeed, the larger m is, the faster the power function blows up as z approaches z_0. If the term in the series that blows up the fastest is a $(z - z_0)^{-N}$ term, then we say that the function has a pole of order N at z_0.

If the Laurent series of $f(z)$ has nonvanishing $(z - z_0)^{-m}$ terms of arbitrarily large m, $f(z)$ is said to have an essential singularity at z_0. An example is the series of (2.24), which has an essential singularity at the origin.

Let us integrate the Laurent series over a closed contour c that encloses z_0 but no other singularities of $f(z)$. Let the direction of c be counterclockwise. By (2.15), all terms in this series except the term

$$a_{-1}/(z - z_0)$$

are integrated to zero. Thus we get

$$\oint_c f(z)dz = 2\pi i a_{-1}. \tag{2.26}$$

This is known as the Cauchy residue theorem. The coefficient a_{-1} is known to be the residue of $f(z)$ at z_0, which we shall denote as $\mathrm{Res}(z_0)$. If the contour is clockwise, the integral will be equal to the negative of $2\pi i$ times the residue.

This formula is one of the most useful formulae in complex analysis. It tells us that the value of an integral over a closed contour can be obtained by simply evaluating the residue of its integrand.

If the contour c encloses more than one singularity of $f(z)$, we replace the right side of (2.26) by the sum of residues of $f(z)$ at these singularities. (Why?)

Before we close this section, let us show how to evaluate efficiently the residue of $f(z)$ at z_0 where the function has a pole of the first order, which is called a simple pole. If the singularity of $f(z)$ at z_0 is a simple pole,

$$f(z) = \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + \cdots .$$

Thus the residue of $f(z)$ at z_0 is equal to

$$\mathrm{Res}(z_0) = \lim_{z \to z_0} (z - z_0) f(z). \tag{2.27a}$$

As an example, let us calculate the residue of $e^z/\sin z$ at $z = 0$. We have by (2.27a) that

$$\mathrm{Res}(0) = \lim_{z \to 0} \frac{ze^z}{\sin z}.$$

We see that both the numerator and the denominator vanish as $z \to 0$; thus we apply l'Hopital's rule and get

$$\mathrm{Res}(0) = \lim_{z \to -0} \frac{e^z + ze^z}{\cos z}.$$

Since ze^z vanishes as $z \to 0$, we have

$$\text{Res}(0) = \lim_{z \to 0} \frac{e^z}{\cos z} = 1.$$

Note that $e^z/\cos z$ is obtained from $e^z/\sin z$ by replacing the denominator of the latter function with the derivative of its denominator. Thus we may dispense with the formula (2.27a) and calculate the residue by differentiating the denominator.

More generally, let $f(z) = g(z)/h(z)$, where $g(z)$ and $h(z)$ are analytic at z_0. If $h(z)$ has a simple zero at z_0, then $f(z)$ has a simple pole at z_0. By (2.27a), we obtain with the use of l'Hopital's rule that the residue of $f(z)$ at z_0 is

$$\text{Res}(z_0) = \frac{g(z_0)}{h'(z_0)}. \tag{2.27b}$$

If $f(z)$ has a double pole at z_0, the Laurent series expansion of $f(z)$ in the neighborhood of z_0 is

$$f(z) = \frac{a_{-2}}{(z - z_0)^2} + \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + \cdots .$$

In this case, the limit on the right side of (2.27a) is equal to infinity, not the residue of $f(z)$ at z_0. To eliminate the singularity at z_0, we multiply $f(z)$ by $(z - z_0)^2$ and get

$$(z - z_0)^2 f(z) = a_{-2} + a_{-1}(z - z_0) + a_0(z - z_0)^2 + \cdots .$$

The expression above is finite in the limit $z \to z_0$, but this limit is equal to a_{-2}, not a_{-1}. To obtain a_{-1}, we differentiate the expression above and then set $z = z_0$. We get

$$a_{-1} = \lim_{z \to z_0} \frac{d}{dz} \left[(z - z_0)^2 f(z) \right].$$

D. Evaluation of Real Integrals

The Cauchy residue theorem provides us with a tool to evaluate a number of integrals in the real world, the integrands of which are functions of a real variable and the integration is over real values of the variable. Some of these integrations are difficult to carry out in closed form with the methods provided by calculus. We shall show that, by going into the never-never land of the complex plane, sometimes we can find the closed forms of these integrals.

As an example, let us consider the integral

$$I = \int_{-\infty}^{\infty} \frac{dx}{1+x^2}. \tag{2.28}$$

While this integral can be evaluated with calculus, we shall use it as an example to demonstrate how to do real integrals with contour integration.

We regard this integral as a contour integral over the real axis of the complex plane. But we cannot as yet apply the Cauchy residue theorem to it, as the real axis is not a closed contour. Let us think of the real axis as the contour from $-R$ to R along the real axis, in the limit as R approaches infinity. We add to this contour the counterclockwise semicircle in the upper half-plane with the origin as the center and R the radius. (See Figure 2.6.) Now we get a closed contour which we shall call c. As we shall see, the integral over the semicircle vanishes in the limit of $R \to \infty$. Thus the integral of (2.28) is equal to the integral over c. Since c is a closed contour we may apply the Cauchy residue theorem to the integral. The only singularity of the integrand enclosed by c is $z = i$. Thus we have by (2.27b),

$$I = 2\pi i \,\mathrm{Res}(i) = 2\pi i \frac{1}{2i} = \pi.$$

To finish the argument let us show that the contribution of the semicircle is zero in the limit $R \to \infty$. If z is a point on the semicircle,

$$z = e^{i\theta} R, \; 0 \le \theta \le \pi.$$

When R is very large, the integrand $1/(1+z^2)$ is approximately equal to $1/z^2$, the magnitude of which is $1/R^2$. We also have

$$dz = ie^{i\theta} R d\theta. \tag{2.29}$$

Thus we have

$$\int_{C_R} \frac{dz}{1+z^2} \approx \int_0^{\pi} \frac{ie\theta R d\theta}{R^2 e^{2i\theta}},$$

where C_R is the semicircle in the upper half-plane. In the limit $R \to \infty$, the integral above vanishes, as there are two factors of R in the denominator of the integrand and only one factor of R in the numerator of the integrand.

We may also close the contour of the integral in (2.28) by adding to it the semicircle in the lower half-plane in the clockwise direction. The only singularity enclosed by this contour is the one at $z = -i$. Thus we have

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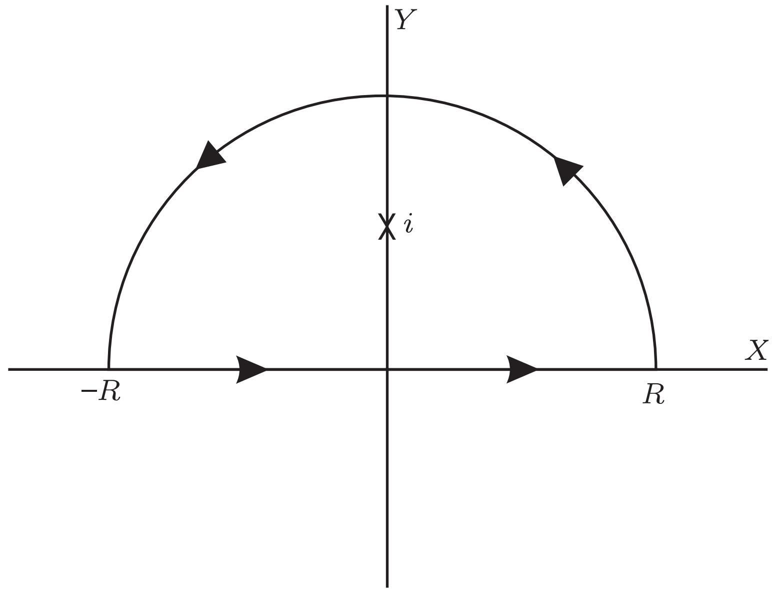

Figure 2.6

$$I = -2\pi i \,\mathrm{Res}(-i) = -2\pi i \frac{1}{-2i} = \pi,$$

which is the same answer. Note that the first minus sign above is due to the fact that the closed contour is clockwise.

One of the first things we do in applying the Cauchy residue theorem is to make sure that the contour is a closed one. If the contour is not closed, try to close it if possible. The second step is to locate the singularities of the integrand enclosed by the contour, and calculate the residues of the integrand at each of the singularities.

Problem for the Reader

Evaluate the integral

$$I = \int_{-\infty}^{\infty} \frac{dx}{(x-i)(x-2i)(x-3i)(x-4i)}.$$

Solution

We may close the contour upstairs by adding to the contour of integration the semicircle in the upper half-plane. This is justified, as the contribution of the semicircle of radius R is of the order of

$$R/R^4,$$

where the numerator factor R comes from dz given by (2.29) and the factor R^4 is the order of the denominator of the integrand when R is large. Thus the contribution of the semicircle vanishes as $R \to \infty$. With the semicircle included, the contour is closed and encloses the four singularities of the integrand located at i, $2i$, $3i$, and $4i$. Therefore, we may obtain the value of I by adding up the residues at these singularities.

It is far simpler, however, to evaluate this integral by closing the contour downstairs. The contribution from the semicircle in the lower half-plane vanishes as before. Since the integrand has no singularities in the lower half-plane, we get

$$I = 0.$$

For the integral

$$I = \int_{-\infty}^{\infty} \frac{dx}{(x+i)(x-2i)(x-3i)(x-4i)},$$

the integrand of which has three poles in the upper half-plane and one pole in the lower half-plane, it is easier if we close the contour downstairs. This is because if we do so, we need to evaluate only the residue at $z = -i$.

For other integrals, there is no way to close the contour before we make some changes. Consider

$$I = \int_{-\infty}^{\infty} \frac{\cos x}{1+x^2} dx. \tag{2.30}$$

We emphasize that it is impossible to close the contour for the integral of (2.30) either upstairs or downstairs. The culprit is the factor $\cos x$ in the numerator of the integrand. While $\cos x$ is of finite values between -1 and 1, no matter how large x is, $\cos z$ is very large when z is complex and large. To see this, we have from (2.22) that

$$\cos z = \frac{e^{ix-y} + e^{-ix+y}}{2}.$$

The first term on the right side of this equation blows up if y goes to $-\infty$, and the second term on the right side of this equation blows up if y goes to ∞. Thus $\cos z$ blows up exponentially when either $y \to \infty$ or $y \to -\infty$. As a result, the contribution from the infinite semicircles, either the one in the upper half-plane or the one in the lower half-plane, is not zero.

Now we note that

$$\cos x = \text{Re}\; e^{ix}.$$

Since $(1 + x^2)$ is real, we have

$$I = \text{Re}\; J,$$

where

$$J = \int_{-\infty}^{\infty} \frac{e^{ix}}{1 + x^2} dx.$$

It is possible to close the contour upstairs for the integral J, as $e^{iz} = e^{ix-y}$ is exponentially small when $y \to \infty$. Therefore, we are allowed to close the contour upstairs. A more detailed discussion of this can be found in Appendix B of Chapter 8. The only singularity of the integrand enclosed by this contour is at $z = i$. We easily get, using (2.27b),

$$J = 2\pi i\, \text{Res}(i) = 2\pi i \frac{e^{iz}}{2z}|_{z=i} = \frac{\pi}{e}.$$

Hence we have

$$I = \int_{-\infty}^{\infty} \frac{\cos x}{1 + x^2} dx = \frac{\pi}{e}. \tag{2.31}$$

Alternately, we may make use of the relation

$$\cos x = \text{Re}\; e^{-ix}$$

and get

$$I = \text{Re}\; K,$$

where

$$K = \int_{-\infty}^{\infty} \frac{e^{-ix}}{1 + x^2} dx.$$

Since $e^{-iz} = e^{-ix+y}$ vanishes as y goes to minus infinity, we may close the contour downstairs for the integral K and get the same answer for I.

Problem for the Reader

Evaluate the integral

$$I = \int_{-\infty}^{\infty} \frac{x \sin x}{1+x^2} dx. \tag{2.32}$$

Solution

Again, the first step is to make the contour a closed contour. If this is done, we may evaluate the integral by the use of Cauchy's residue theorem.

But it is not possible to close the contour either upstairs or downstairs with the factor $\sin x$ in the numerator. This is because, just like the magnitude of $\cos z$, the magnitude of $\sin z$ is exponentially large if y, the imaginary part of z, is large, regardless of whether y is positive or negative.

Let us try another way. We have

$$\sin x = \mathrm{Im} e^{ix},$$

and since the other factors of the integrand are real, we get

$$I = \mathrm{Im} J,$$

where

$$J = \int_{-\infty}^{\infty} \frac{x e^{ix}}{1+x^2} dx.$$

We close the contour upstairs and get, using (2.27b),

$$J = 2\pi i \frac{ie^{-1}}{2i} = \frac{\pi i}{e}.$$

Thus

$$I = \int_{-\infty}^{\infty} \frac{x \sin x}{1+x^2} dx = \frac{\pi}{e}. \tag{2.33}$$

Next we consider the integral

$$I = \int_{-\infty}^{\infty} \frac{\sin x}{x} dx. \tag{2.34}$$

Note that, although its denominator vanishes at $x = 0$, $\sin x/x$ is finite at $x = 0$, as its numerator vanishes at $x = 0$ as well. Indeed, $\sin z/z$ is an entire function of z.

Again, it is not possible to close the contour of integration with $\sin x$ as the numerator. So let us try the trick of replacing I by the imaginary part of J, where

$$J = \int_{-\infty}^{\infty} \frac{e^{ix}}{x} dx.$$

But the integrand of J blows up at $x = 0$. This is because the denominator of the integrand vanishes at $x = 0$, while the numerator of the integrand does not. Therefore, the integral J is divergent and meaningless, and the trick of replacing $\sin x$ with e^{ix} fails.

Since the origin is a troublesome point, let us deform the contour away from the origin. This is possible as the integrand $\sin z/z$ is analytic at $z = 0$. It does not matter what precisely the contour is. We may, for example, deform the contour into c, where c goes from $-\infty$ to -1 along the real axis, from -1 to 1 along a curve lying in the upper half-plane, and from 1 to ∞ along the real axis. Thus we have

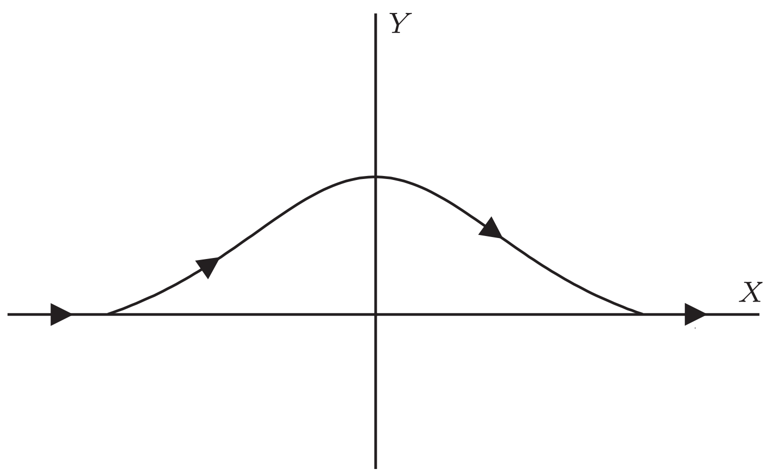

Figure 2.7

$$I = \int_{c} \frac{\sin z}{z} dz. \tag{2.35}$$

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By (2.23) we have

$$I = I_1 + I_2,$$

where

$$I_1 = \int_c \frac{e^{iz}}{2iz} dz, \tag{2.36}$$

and

$$I_2 = -\int_c \frac{e^{-iz}}{2iz} dz. \tag{2.37}$$

The numerator of the integrand of I_1 is e^{iz}, which allows us to close the contour upstairs. Since the integrand of I_1 is analytic in the upper half-plane, we have

$$I_1 = 0.$$

For the integral I_2 we close the contour downstairs. Since the contour is clockwise and since the only singularity of the integrand enclosed by the contour is located at $z = 0$, we get, using (2.27b),

$$I_2 = (-2\pi i)\left(-\frac{1}{2i}\right) = \pi.$$

Thus we find

$$I = \int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \pi. \tag{2.38}$$

Had we not deformed the contour away from the origin, both I_1 and I_2 would have been divergent integrals, as the integrands of these integrals blow up at the origin.

The contour of integration for an integral is not always the entire real axis, and it is not always possible to close the contour either upstairs or downstairs. Nevertheless, sometimes we are still able to do so after making some minor changes. As an example, consider the integral

$$\int_0^{\infty} \frac{dx}{1 + x^4}, \tag{2.39}$$

the contour of integration of which is the positive real axis only. Now the integrand of this integral is an even function of x. This is to say that the integrand at $x = -r$

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on the negative real axis is equal to the integrand at $x = r$ on the positive real axis. Thus we may integrate over both the positive real axis and the negative real axis and divide the result by two. Therefore, we have

$$\int_0^\infty \frac{dx}{1+x^4} = \frac{1}{2}\int_{-\infty}^\infty \frac{dx}{1+x^4}. \tag{2.40}$$

Now that the contour of integration is the entire real axis, we may evaluate the integral by closing the contour either upstairs or downstairs. Let us close the contour in the upper half-plane. The singularities of the integrand enclosed are $e^{i\pi/4}$ and $e^{3i\pi/4}$. We get, after adding up the residues of these two points,

$$\int_0^\infty \frac{dx}{1+x^4} = \frac{\sqrt{2}\pi}{4}. \tag{2.41}$$

Consider next the integral

$$I = \int_0^\infty \frac{dx}{1+x^5}. \tag{2.42}$$

Since the integrand is not an even function of x, it will not help if we add the negative real axis to the contour of integration and divide by two. Instead, let us consider the integrand on the ray of argument $2\pi/5$ in the complex plane, i.e.,

$$z = re^{2\pi i/5}. \tag{2.43}$$

On this ray,

$$z^5 = r^5.$$

Therefore, the integrand at $z = re^{2\pi i/5}$ is $1/(1+r^5)$, which is the same as the integrand at $z = r$ on the positive real axis.

Let us therefore consider the integral

$$J = \oint_c \frac{dz}{1+z^5}, \tag{2.44}$$

where c is the closed contour consisting of the ray that is the positive real axis going from zero to R, the ray that given by (2.43) going from $r = R$ to $r = 0$, and the arc that is of the distance R from the origin and joins these two rays. We shall take the limit $R \to \infty$ at the end.

On the arc, the integrand vanishes like R^{-5} as $R \to \infty$. On the other hand, the length of the arc is $2\pi R/5$, which is merely linearly proportional to R. Thus

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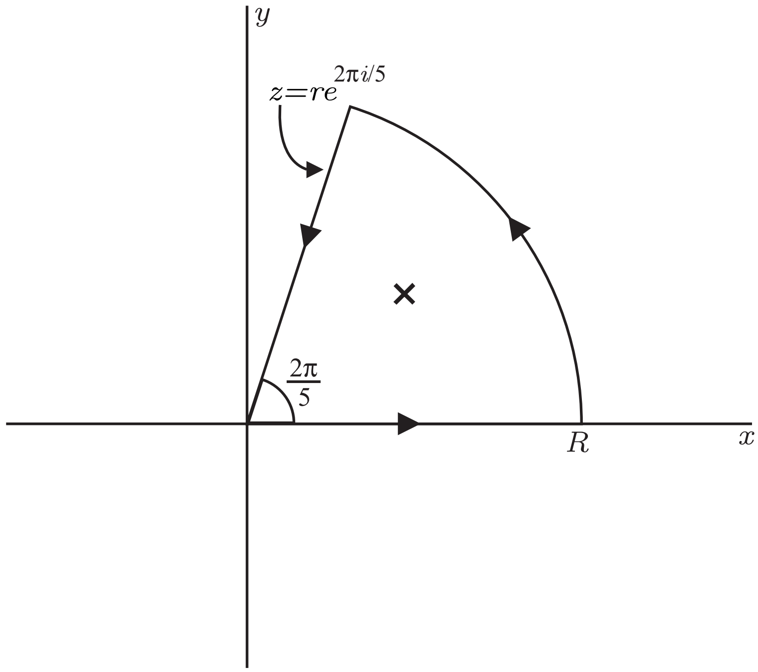

Figure 2.8

the integrand times the length of the arc vanishes as $R \to \infty$. Therefore, the contribution from the arc is zero as $R \to \infty$.

The integral over the ray of (2.43) in the limit $R \to \infty$ is related to I of (2.42) by

$$\int_{\infty}^{0} \frac{e^{2\pi i/5} dr}{1 + r^5} = -e^{2\pi i/5} I.$$

Therefore, we have

$$J = (1 - e^{2\pi i/5}) I. \tag{2.45}$$

We may evaluate J with the Cauchy residue theorem. The singularities of the integrand are located at

$$e^{i(2n+1)\pi/5}. \tag{2.46}$$

The only singularity enclosed by c is $e^{i\pi/5}$. Thus we have, by (2.27b),

$$J = \frac{2\pi i}{5 e^{4i\pi/5}}. \tag{2.47}$$

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Hence we get

$$\int_0^\infty \frac{dx}{1+x^5} = \frac{J}{1-e^{2\pi i/5}} = \frac{\pi}{5\sin\pi/5}. \tag{2.48}$$

The exact value of $\sin\pi/5$ can be deduced from the result in homework problem 2.

As the final example of contour integration, we evaluate the integral

$$I = \int_0^{2\pi} \frac{d\theta}{a+b\cos\theta}, \tag{2.49}$$

where a and b are positive. We require that $a > b$ so that the denominator of the integrand does not vanish for any θ.

While the contour of integration is the finite interval $[0, 2\pi]$, which is not closed, we may transform it into a closed contour by making a change of variable. We put

$$z \equiv e^{i\theta}.$$

As θ varies from 0 to 2π, z traverses in the counterclockwise direction the unit circle with the center at the origin. This circle is a closed contour. We have

$$dz = e^{i\theta} i d\theta,$$

or

$$d\theta = \frac{dz}{iz}.$$

We also have

$$\cos\theta = \frac{1}{2}(z+z^{-1}).$$

Thus

$$I = \oint_{c_1} \frac{2dz}{ib(z^2+2az/b+1)},$$

where c_1 is the unit circle with center at the origin. The singularities of the integrand are located at the zeroes of the denominator of the integrand, which are

$$z = -a/b \pm \sqrt{a^2/b^2-1}.$$

The singularity enclosed by c_1 is the one above with the plus sign. Applying the Cauchy residue theorem, we get, by (2.27b),

$$\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta} = \frac{2\pi}{\sqrt{a^2 - b^2}}. \tag{2.50}$$

E. Branch Points and Branch Cuts

Consider the function

$$\log z = \log(re^{i\theta}) = \ln r + i\theta. \tag{2.51}$$

Let us start out at the point A in the figure below, follow the closed path c_1 in the figure in the counterclockwise sense, and come back to point A.

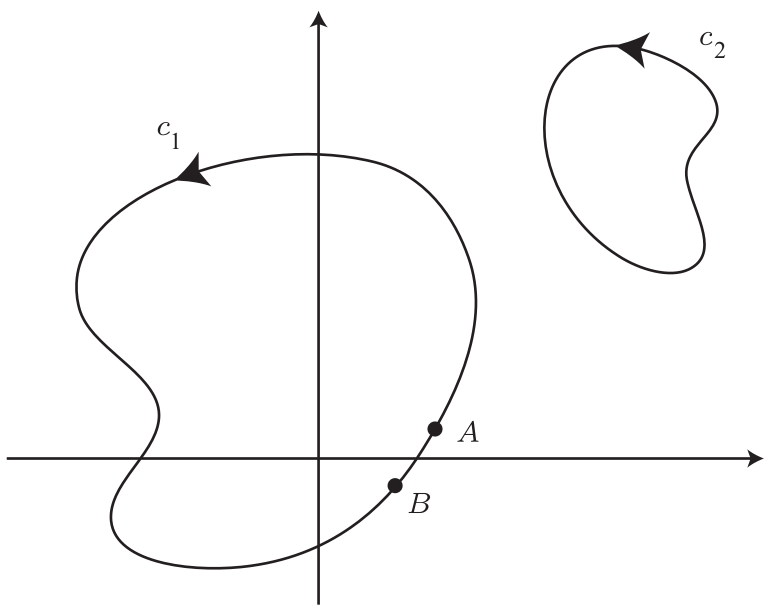

Figure 2.9

It is clear that while we return to the same location, the value of θ is not the same anymore. Indeed it changes to $\theta + 2\pi$. Therefore, upon traversing the closed path, we have

$$\log z \to \log z + 2\pi i,$$

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which means that $\log z$ does not come back to its original value. This is true as long as one defines it to be a continuous function along the closed path.

Note that if we follow the same closed path around the origin in the clockwise sense and return to the same point, the value of θ changes to $\theta - 2\pi$. Thus the function $\log z$ changes to $(\log z - 2\pi i)$. And if the closed path goes around the origin in the counterclockwise sense n times, $\log z$ changes to $\log z + 2n\pi i$.

Thus we have an identity crisis: Which value should we choose to be the value of $\log z$ at A?

It may help us to find the answer to this question if we repeat the considerations above with the curve c_2 in Figure 2.9. We note that the value of θ does not change as one starts at a point on c_2 in the figure, traverses the closed path and comes back to the starting point. As a consequence, the value of $\log z$ does not change after the closed path c_2 is traversed.

What is the difference between the paths c_1 and c_2? The answer is that c_1 encloses the origin, while c_2 does not. Since the polar angle θ is the angle of the position vector joining the origin to z, θ increases by 2π as one goes once around c_1, and does not change as one goes around c_2. As a consequence, the function $\log z$ changes its value if the traversed path is c_1, which encloses the origin, but does not change its value if the traversed path is c_2, which does not enclose the origin. The origin is a special point with respect to the function $\log z$; it is said to be a branch point of the function $\log z$.

More generally, we define the point z_0 as a branch point of the function $f(z)$ if $f(z)$ changes its value as one traces a closed path enclosing z_0.

If we restrict ourselves to an open region that does not include the origin as an interior point, the function $\log z$ can be uniquely defined in this region. This is because there is no closed path inside the region that encloses the origin. Therefore, we may choose a point inside this region and define the value of θ at this point to be between 0 and 2π, say. Then the value of θ for any point inside this region is uniquely defined, and so is the function $\log z$. The designation of the value of θ at the chosen point is not unique. For example, we may define θ at the chosen point to be between 2π and 4π, or between $-\pi$ and π. The values of $\log z$ in this region are different with different definitions, but the function $\log z$ is single-valued with each designation of the value of θ at the chosen point. There is, therefore, more than one consistent definition of $\log z$ in a region. They are called different branches of $\log z$.

We may verify that the real part and the imaginary part of $\log z$ satisfy the Cauchy-Riemann equations everywhere except at the origin. Then $\log z$ is analytic in a region in which a branch of $\log z$ is chosen and the function is uniquely defined.

Let us cut up the complex z-plane by drawing a curve joining the origin to infinity. To be specific, we shall choose this curve to be the positive real axis. In the cut z-plane there is no closed contour that encloses the origin. We call the positive real axis a branch cut of $\log z$.

Let us start at point A, which is above and infinitesimally close to the positive real axis, and go along a circle in the counterclockwise sense, arriving at point B, which is below and infinitesimally close to the positive real axis. Then the value of $\log z$ at B differs from that at A by $2\pi i$. Thus $\log z$ is discontinuous across the positive x axis. Nevertheless, if we restrict ourselves in the cut z-plane, the function is single-valued. This is because the points A and B, separated by the branch cut, are not regarded as the same point.

This is like thinking of $f(z)$ as a function not on a plane but on a parking garage that has many levels. Let the point A be the entry point of the garage. Let us choose the value of θ at this point to be zero. As we go around a full circle in the counterclockwise sense, we arrive not at the entry point of the garage but at the point one level above it. The value of $\log z$ is taken to be dependent on which level we are at. Therefore, while the value of $\log z$ changes by $2\pi i$ as we go up one level, the function $\log z$ is uniquely defined at each point of the parking garage.

If we go around the garage in the counterclockwise sense n times, we arrive at the $(n+1)^{\text{th}}$ level of the garage. This level is called the $(n+1)^{\text{th}}$ Riemann sheet of the function. Since n can be any positive or negative integer, $\log z$ has infinitely many Riemann sheets.

There is no reason to restrict the branch cut to be on the positive real axis. We may choose the branch cut to be on the negative real axis. If we start at the entry point A with $\theta = 0$ as before, then we are choosing the branch of the function on the garage that is half a level below ground and half a level above ground. The values of θ in this branch are between $-\pi$ and π. We may, indeed, choose the branch cut to be any curve joining the origin to infinity in any way.

While the point $z = 0$ is a singularity of $\log z$ in the finite z-plane, the point infinity is also a branch point of $\log z$. To see this, we put

$$z = \frac{1}{\omega}.$$

Then the point $\omega = 0$ corresponds to the point $z = \infty$. Since

$$\log z = -\log \omega,$$

the function $\log z$ has a branch point at $\omega = 0$, or $z = \infty$. We also realize that a branch cut of $\log z$, chosen in any way we just described, is always a curve joining the only two branch points of the function.

It is straightforward to find the branch points for the function $\log(z - z_0)$. Let $\omega = z - z_0$; then

$$\log(z - z_0) = \log \omega.$$

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Since $\omega = 0$ and $\omega = \infty$ are the branch points of the function $\log \omega$, $z = z_0$ and $z = \infty$ are the branch points of $\log(z - z_0)$.

Problem for the Reader

Find the branch points of $\log(z^2 - 1)$. Draw some possible sets of branch cuts.

Solution

We have
$$\log(z^2 - 1) = \log(z - 1) + \log(z + 1).$$
Thus the points -1, 1, and ∞ are the branch points of the function $\log(z^2 - 1)$.

We shall draw branch cuts to ensure the function is single-valued in the cut plane. Some possible sets of branch cuts for the function $\log(z^2 - 1)$ are drawn below.

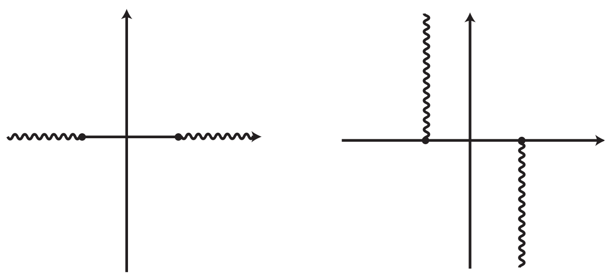

Figure 2.10

Problem for the Reader

Find the branch points of the function
$$\log\left(\frac{z - i}{z + i}\right).$$
Draw two sets of branch cuts for this function.

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Solution

Since

$$\log\left(\frac{z-i}{z+i}\right) = \log(z-i) - \log(z+i),$$

the branch points of the function in the finite plane are i and $-i$.

We draw two possible sets of branch cuts in the figure below. While the set of branch cuts in the right-hand figure is self-explanatory, we shall say a few words about the left-hand figure. Let us traverse in the counterclockwise sense a closed path enclosing both branch points. Since this closed path encloses the point i, we have, as we return to the starting point,

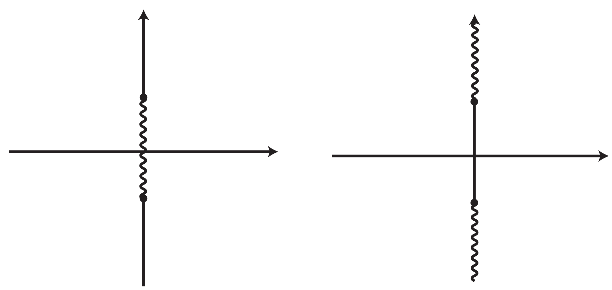

Figure 2.11

$$\log(z-i) \to \log(z-i) + 2\pi i.$$

And since this closed path also encloses the point $-i$, we have

$$\log(z+i) \to \log(z+i) + 2\pi i.$$

Since

$$\log\left(\frac{z-i}{z+i}\right)$$

is equal to the difference of $\log(z-i)$ and $\log(z+i)$, its value does not change as the closed path is traversed. Therefore, it suffices to draw just one finite branch cut between i and $-i$ to make the function

$$\log\left(\frac{z-i}{z+i}\right)$$

single-valued in the cut plane. This is because there is no closed path in this cut plane, which encloses just one of the branch points.

Note that this branch cut is a finite branch cut. That this is possible means that infinity is not a branch point. This is easily verified directly. Let $z = \omega^{-1}$; then

$$\log\left(\frac{z-i}{z+i}\right) = \log\left(\frac{1-i\omega}{1+i\omega}\right).$$

We see from the right side of the equation above that $\omega = 0$ is not a branch point of the function.

Since infinity is not a branch point, it is not necessary to draw a branch cut joining infinity with a finite point. We may therefore think of the two branch cuts in the right-hand figure above as joining with each other at infinity, forming just one continuous branch cut. We may draw this branch cut by starting from $-i$ and moving downward along the negative imaginary axis, passing through infinity and following the positive imaginary axis to the point i. This is somewhat like the way Columbus tried to get to India.

Next we discuss the function

$$z^a = r^a e^{ia\theta},$$

where a is a complex number. As we traverse a closed path enclosing the origin once in the counterclockwise sense, we have

$$z^a \to e^{i2\pi a} z^a.$$

Since the value of z^a changes after such a trip, the origin is a branch point of z^a. Let $z = \omega^{-1}$. Then we have $z^a = \omega^{-a}$, and we conclude that the point $z = \infty$ is also a branch point of z^a. Indeed, the points 0 and ∞ are the only two branch points of the function z^a. Thus this function is singled-valued in the plane with a cut joining the origin with infinity.

We call attention to the fact if $a = n$, where n is an integer of either sign, then $e^{i2\pi a}$ is equal to $e^{i2\pi n}$, which is unity. Thus the value of z^n does not change after a closed path enclosing the origin is traversed. Therefore, $z = 0$ and $z = \infty$ are not branch points for the function z^n, where n is either a positive integer or a negative integer.

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Next we consider the case when a is equal to the rational number m/n, where m and n are integers having no common factors. The value of the function $z^{m/n}$ changes by a multiple of $e^{i2\pi m/n}$ after a path enclosing the origin is traversed once in the counterclockwise sense. Thus the origin is a branch point of the function $z^{m/n}$. But after we go around the origin in the counterclockwise sense n times, the function changes by a multiple of $e^{i2\pi m}$, which is equal to unity. This says that the function $z^{m/n}$ has only n Riemann sheets.

Problem for the Reader

Find the branch points of the function $z^a(1-z)^b$.

Solution

If neither a nor b is an integer, the function has branch points at $z = 0$ and $z = 1$ in the finite plane.

Let $z = \omega^{-1}$; then

$$z^a(1-z)^b = \omega^{-a-b}(\omega - 1)^b.$$

The function has a branch point at $\omega = 0$ unless $(a + b)$ is an integer. Thus $z^a(1-z)^b$ has a branch point at $z = \infty$ unless $(a + b)$ is an integer.

As an example, consider the function $(z-i)^{-1/2}(1-z)^{3/2}$. The points i and 1 are the branch points of this function, but infinity is not. Therefore, this function is single-valued in the plane with a finite branch cut connecting i to 1.

We give a few pointers below to help the reader find the branch points of a function.

a. If z_0 is a branch point of $f(z)$, it is also a branch point of $\log f(z)$. This is because as the value of $f(z)$ changes, so does the value of $\log f(z)$.

Similarly, a branch point of $f(z)$ is a possible branch point of $[f(z)]^a$. We qualify with the word "possible," as there are exceptions. An example is $f(z) = \sqrt{z}$ and $a = 2$. In this case, the origin is a branch point for $f(z)$, but not that of $[f(z)]^2$.

b. If z_0 is a zero of $f(z)$, it is a branch point of $\log f(z)$. It is also a possible zero of $[f(z)]^a$. To prove this, let z_0 be an n^{th}-order zero of $f(z)$, i.e.,

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$$f(z) = (z - z_0)^n F(z),$$

where $F(z)$ is analytic at z_0 and $F(z_0) \neq 0$. Then

$$\log f(z) = \log F(z) + n \log(z - z_0).$$

We see from the formula above that z_0 is a branch point of $\log f(z)$. Also, we have

$$[f(z)]^a = (z - z_0)^{na} [F(z)]^a.$$

From the formula above, we see that z_0 is a branch point of $[f(z)]^a$ unless na is an integer.

c. If z_0 is a pole of $f(z)$, it is a branch point of $\log f(z)$. It is also a possible branch point of $[f(z)]^a$. To prove this, let z_0 be an n^{th}-order pole of $f(z)$, i.e.,

$$f(z) = \frac{F(z)}{(z - z_0)^n},$$

where $F(z)$ is analytic at z_0 and $F(z_0) \neq 0$. Thus

$$\log f(z) = \log F(z) - n \log(z - z_0).$$

We see from the formula above that z_0 is a branch point of $\log f(z)$.

Also, we have

$$[f(z)]^a = (z - z_0)^{-na} [F(z)]^a.$$

From the formula above, we see that z_0 is a branch point of $[f(z)]^a$ unless na is an integer.

Therefore, to locate the branch points of $\log f(z)$ or $[f(z)]^a$, we look for the zeroes, the poles, and the branch points of $f(z)$.

Problem for the Reader

Find the branch points for $\left[1 - (1 - z^2)^{1/2}\right]^{1/3}$.

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Solution

Let $f(z) = 1 - (1 - z^2)^{1/2}$. Since $z = \pm 1$ are the branch points of $f(z)$, by point (a) they are also branch points of the function $[f(z)]^{1/3}$. To find the zeroes of $1 - (1 - z^2)^{1/2}$, or the roots of the equation

$$1 = (1 - z^2)^{1/2},$$

we square the equation above and get

$$z = 0.$$

But squaring an equation may produce roots that are not the roots of the equation. So let us see if $z = 0$ satisfies $\sqrt{1 - z^2} = 1$. We have

$$\sqrt{(1 - z^2)}|_{z=0} = \pm 1.$$

Thus the equation is not satisfied unless we choose the value of $\sqrt{1 - z^2}|_{z=0}$ to be 1. To say this in another way, whether $z = 0$ is a branch point for the function $[f(z)]^{1/3}$ depends on the branch we choose for the function $\sqrt{1 - z^2}$. And if we choose the branch so that the function $\sqrt{1 - z^2}$ at $z = 0$ is unity, then for z very small, we have

$$\sqrt{1 - z^2} \simeq 1 - z^2/2.$$

Thus

$$[f(z)]^{1/3} \simeq z^{2/3}/2^{1/3},$$

and $z = 0$ is a cubic-root branch point of the function.

To see whether infinity is a branch point for $[f(z)]^{1/3}$, we set $z = \omega^{-1}$, and we get

$$[f(z)]^{1/3} = \left[\omega - \sqrt{\omega^2 - 1}\right]^{1/3} \omega^{-1/3}.$$

Therefore, $z = \infty$ is a cubic-root branch point for the function $\left[1 - (1 - z^2)^{1/2}\right]^{1/3}$.

In summary, the branch points of the function in the z-plane are ± 1, ∞. In addition, the origin is a branch point of the function provided that we choose the branch in which $\sqrt{1 - z^2}$ is equal to unity at $z = 0$.

Problem for the Reader

Find the branch points of the function $\log\left(z+\sqrt{1-z^2}\right)$.

Solution

Let $f(z) = \left(z+\sqrt{1-z^2}\right)$. Since the points ± 1 are the branch points of $f(z)$, they are the branch points of the function $\log f(z)$.

Next, the zeroes of $f(z)$ are given by

$$\sqrt{1-z^2} = -z.$$

There is no root for this equation.

Let $z = \omega^{-1}$, and we get

$$\log\left(z+\sqrt{1-z^2}\right) = \log\left(\frac{1+\sqrt{\omega^2-1}}{\omega}\right).$$

We see from the formula above that $\omega = 0$ or $z = \infty$ is a logarithmic branch point of the function.

To summarize, the branch points for the function in the z-plane are ± 1 and ∞.

We'll give an example to illustrate how to use the concept of branch cuts to calculate a real integral.

Example:

Evaluate the integral

$$I = \int_0^\infty \frac{\ln x}{4+x^2} dx$$

with contour integration.

The first thing to do is to relate this integral with another integral the contour of integration of which is a closed contour. We apply Cauchy's residue theorem to evaluate the latter, and obtain the value of the former after the value of the latter is found.

We draw a branch cut joining the origin to $-i\infty$, and choose θ, the argument of z, to be zero on the positive real axis. The function $\log z$ on this Riemann sheet is then uniquely defined. In particular, $\log z$ is equal to $\ln x$ on the positive real axis. Therefore, we may express I as $\log z/(4+z^2)$ integrated over the positive real axis. This contour of I is not closed.

Let us define another integral

$$J \equiv \int_{-\infty}^{\infty} \frac{\log z}{4+z^2} dz.$$

The contour of J can be closed upstairs, as the contribution of the infinite semicircle in the upper half-plane is of the order of the limit of $(R\ln R)/R^2$ as $R \to \infty$, where the factor $\ln R$ in the numerator of this quantity comes from the numerator of the integrand, the factor R^2 in the denominator of this quantity comes from the denominator of the integrand, and the factor R in the numerator of this quantity is the order of magnitude of the length of the arc. This quantity is equal to zero in the limit $R \to \infty$.

The only singularity enclosed by this closed contour is at $z = 2i$. Note that, starting at the point 2, we may reach the point $2i$ by traversing the counterclockwise circular arc of radius 2 with angular width $\pi/2$. Thus the argument of $2i$ is $\pi/2$, and we have

$$2i = 2e^{i\pi/2}.$$

As a result,

$$J = 2\pi i \frac{\log(2e^{i\pi/2})}{4i} = \pi \frac{\ln 2 + i\pi/2}{2}.$$

To obtain I from J, we express J as

$$J = J_1 + J_2,$$

where

$$J_1 = \int_0^{\infty} \frac{\log z}{4+z^2} dz,$$

and

$$J_2 = \int_{-\infty}^{0} \frac{\log z}{4+z^2} dz.$$

First of all, J_1 is simply I. Next we shall show that J_2 is related to I as well. We have, on the negative real axis,

$$\log z = \log(re^{i\pi}) = \ln r + i\pi.$$

Thus

$$J_2 = \int_0^\infty \frac{i\pi + \ln r}{4 + r^2} dr = \frac{i\pi^2}{4} + I.$$

Therefore,

$$J = 2I + \frac{i\pi^2}{4}.$$

With the value of J obtained earlier with the help of the Cauchy residue theorem, we get

$$I = \int_0^\infty \frac{\ln x}{4 + x^2} dx = \pi \frac{\ln 2}{4}. \tag{2.52}$$

We close this section with two topics: (a) a discussion of the principal value of an integral, and (b) a discussion of the Plemelj formulae.

Back in the high school days when we were first introduced to integrations over a real variable, some of us might have puzzled over the value of real integrals such as

$$\int_{-2}^{3} \frac{dx}{x}.$$

For, if we carry out the integration in a straightforward way, we get

$$\int_{-2}^{3} \frac{dx}{x} = \ln x \,|_{-2}^{3} = \ln 3 - \ln(-2).$$

But what is the value of $\ln(-2)$?

This ambiguity is due to a difficulty with this integral. The fact is that the integrand $\frac{1}{x}$ has a simple pole at the origin. Since the contour of integration passes through the origin, the integral

$$\int_{-2}^{3} \frac{dx}{x}$$

is actually undefined as it is.

Let us try to define this integral by identifying it with

$$\int_{-2}^{-\epsilon_1} \frac{dx}{x} + \int_{\epsilon_2}^{3} \frac{dx}{x},$$

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where ϵ_1 and ϵ_2 are positive and infinitesimally small constants. This means that we first integrate from $x = -2$ to the point $-\epsilon_1$, which is to the left of the origin, jump over the origin, and continue to integrate from the point ϵ_2, which is to the right of the origin, to the point $x = 3$. In this way we make sure that the contour of integration does not pass through the origin. We define

$$\int_{-2}^{3} \frac{dx}{x}$$

to be this integral in the limit ϵ_1 and ϵ_2 go to zero.

There is something encouraging about this defintion of the divergent integral

$$\int_{-2}^{3} \frac{dx}{x}.$$

For, while both integrals are infinite as ϵ_1 and ϵ_2 go to zero, the first integral is negative and the second integral is positive. Indeed, the integrands of both integrals are the same function $1/x$, which is an odd function of x. Thus there is cancellation between these two integrals, and we hope that the sum of them is finite as ϵ_1 and ϵ_2 go to zero.

To see if this true, we calculate these integrals explicitly. We have

$$\int_{-2}^{-\epsilon_1} \frac{dx}{x} = \int_{\epsilon_1}^{2} \frac{dx}{x} = \ln \frac{\epsilon_1}{2},$$

and

$$\int_{\epsilon_2}^{3} \frac{dx}{x} = \ln \frac{3}{\epsilon_2}.$$

Thus the sum of the two integrals above is

$$\ln \frac{3\epsilon_1}{2\epsilon_2}.$$

This value depends on the ratio

$$\frac{\epsilon_1}{\epsilon_2}$$

and is not unique.

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Let us define the principal value of the divergent integral

$$\int_{-2}^{3} \frac{dx}{x}$$

to be the one with $\epsilon_1 = \epsilon_2$:

$$P\int_{-2}^{3} \frac{dx}{x} = \int_{-2}^{-\epsilon} \frac{dx}{x} + \int_{\epsilon}^{3} \frac{dx}{x}$$

with ϵ infinitesimal. We then have

$$P\int_{-2}^{3} \frac{dx}{x} = \ln 3/2,$$

which is unique.

For some integrals, the singularity of the integrand may be located at x_0 rather than at the origin. Let $f(x)$ blow up like a simple pole at x_0, an interior point of the interval of integration. We define the principal value of the divergent integral $\int_a^b f(x)dx$ to be

$$P\int_{a}^{b} f(x)dx = \int_{a}^{x_0-\epsilon} f(x)dx + \int_{x_0+\epsilon}^{b} f(x)dx,$$

where ϵ is positive and infinitesimal.

We mention that not all divergent integrals can be made finite and unique as we take their principal values. For example, the principal value of the divergent integral

$$\int_{-1}^{2} \frac{dx}{x^2}$$

remains divergent, as $1/x^2$ is positive at the two sides of the origin and there is no cancellation between the integral over positive values of x and that over negative values of x. On the other hand, the principal value of a convergent integral such as

$$\int_{-1}^{2} \frac{\sin x}{x} dx$$

is equal to the integral itself. This is because omitting an infinitesimally small interval of integration does not change the value of a convergent integral.

We also mention that there are other ways to define divergent integrals such as

$$\int_{-2}^{2} \frac{dx}{x}.$$

As an example, the contour of this integral may be chosen to be the half-circle of radius 2 which has the origin as its center and lies in the upper half-plane. We may also make other choices of the contour of integration, but as long as the curve does not pass through the origin, the integral is defined. We may then carry out the integration explicitly and get

$$\int_{-2}^{2} \frac{dz}{z} = \ln \frac{2}{-2}.$$

The value of the integral depends on the difference of the phase of the upper endpoint $z = 2$ and that of the lower endpoint $z = -2$ as the contour is traversed. Let us choose this contour to lie entirely in the upper half-plane, the endpoints being excluded. While there are many such contours, by Cauchy's integral theorem the value of the integral is independent of these contours we choose. Now as we go from -2 to 2 on such a contour, we go clockwise and the phase of the point $z = 2$ is smaller than that of the point $z = -2$ by π. Thus we have, if the entire contour of integration lies above the origin,

$$\int_{-2}^{2} \frac{dz}{z} = \ln\left(\frac{2}{-2}\right) = -i\pi.$$

By choosing the contour to be the one that is infinitesimally above the real axis, we may express this result as

$$\int_{-2}^{2} \frac{dx}{x + i\epsilon} - P\int_{-2}^{2} \frac{dx}{x} = -i\pi,$$

the second integral above being zero as its integrand is an odd function of x.

Problem for the Reader

If we define the contour for the integral

$$\int_{-2}^{3} \frac{dz}{z}$$

to be a curve joining -2 to 3 and lying entirely in the lower half-plane, the endpoints being excluded, find the value of this integral.

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Solution

If we follow such a curve from -2 to 3, we go counterclockwise, and the phase of the point $z = 3$ is larger than that of the point $z = -2$ by π. Thus for all such contours, we have

$$\int_{-2}^{3} \frac{dz}{z} = \ln(3/2) + i\pi,$$

which differs from the principal value of this integral by $i\pi$.

By choosing the contour to be infinitesimally below the real axis, we may express this result as

$$\int_{-2}^{3} \frac{dx}{x - i\epsilon} - P\int_{-2}^{3} \frac{dx}{x} = i\pi.$$

We are now ready to prove the Plemelj formulae, which say that

$$\int_{a}^{b} \frac{r(x')}{x' - (x \pm i\epsilon)} dx = P\int_{a}^{b} \frac{r(x')}{x' - x} dx \pm i\pi r(x)$$

where $a < x < b$. We assume that the function $r(x')$ is such that the integral above is convergent. This does not exclude the possibility for $r(x')$ to be infinite at some point inside the interval of integration. It just means that $r(x')$ must not blow up too badly at any such points for the integral to diverge. For example, it must not blow up as fast as a simple pole at the endpoints a and b.

We also point out that considered as a function of x', the integrand above has a pole at $x \pm i\epsilon$. Since the contour of integration is on the real axis, and since ϵ is infinitesimal, the plus and minus signs in front of ϵ in the integral merely signify whether the pole is above or below the contour of integration.

We write

$$\int_{a}^{b} \frac{r(x')}{x' - x - i\epsilon} dx' = \int_{a}^{b} \frac{r(x') - r(x)}{x' - x - i\epsilon} dx' + \int_{a}^{b} \frac{r(x)}{x' - x - i\epsilon} dx'.$$

Note that the factor $r(x)$ in the last integral above , being independent of x^1, may be placed at the left of the integral sign. We also note that the integrand of the first integral on the right side of the equation above is finite in the limit as ϵ goes to zero. This is because the numerator of the integrand vanishes at $x' = x$. Thus we may ignore the term $-i\epsilon$ in the denominator of this integral. Since the principal value of such an integral is equal to the integral itself, we have

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$$\int_a^b \frac{r(x')-r(x)}{x'-x}dx' = P\int_a^b \frac{r(x')-r(x)}{x'-x-i\epsilon}dx' = P\int_a^b \frac{r(x')}{x'-x}dx' - P\int_a^b \frac{r(x)}{x'-x}dx'.$$

Again, $r(x)$ in the last integral above may be taken to the left of the integral sign. Generalizing slightly what we have discussed, it is possible to prove that

$$\int_a^b \frac{dx'}{x'-x-i\epsilon} - P\int_a^b \frac{dx'}{x'-x} = i\pi.$$

With all these considerations, the first of the Plemelj formulae is obtained.

The second Plemelj formula is obtained from the first Plemelj formula by taking complex conjugation.

We mention that if we replace, in the integral of the Plemelj formula, the real variable x by the complex variable z, we get the function

$$f(z) = \int_a^b \frac{r(x')}{x'-z}dx',$$

which is analytic provided that z is not a point on the interval $[a, b]$. This is because the values of x', the variable of integration, are restricted to the real values between a and b. Therefore, the denominator $(x' - z)$ never vanishes as long as z is not equal to some real value between a and b. The factor $1/(x' - z)$ in the integrand is thus an analytic function of z provided that z is not a point in the interval $[a, b]$, and hence so is $f(z)$.

The function $f(z)$ is discontinuous across the interval $[a, b]$, as we shall presently see.

Problem for the Reader

Find the values of $f(x + i\epsilon)$ and $f(x - i\epsilon)$ when $a < x < b$. Find also the discontinuity of $f(z)$ across the branch cut $a < x < b$.

Solution

By the first of the Plemelj formulae, we have

$$f(x+i\epsilon) = P\int_a^b \frac{r(x')}{x'-x}dx' + i\pi r(x).$$

By the second Plemelj formula, we have

$$f(x-i\epsilon) = P\int_a^b \frac{r(x')}{x'-x}dx' - i\pi r(x).$$

Therefore,

$$f(x+i\epsilon) - f(x-i\epsilon) = 2i\pi r(x)$$

is the discontinuity of $f(z)$ across the branch cut $a < x < b$.

Thus the function $f(z)$ defined by the integral above has a branch cut from a to b, and is analytic everywhere else. We also note that, since $1/(x'-z)$ vanishes as z goes to infinity, $f(z)$ vanishes as z goes to infinity.

The converse is also true: If $f(z)$ is analytic in the complex plane with the exception of a branch cut from a to b on the real axis, does not blow up as fast as a pole either at a or at b, and vanishes at infinity, then $f(z)$ is given by the integral above, with $2\pi i r(x)$ the discontinuity of $f(z)$ across the branch cut. (See homework problem 15.)

This means that we can construct the function $f(z)$ by merely knowing its discontinuity across the branch cut, provided that all the conditions mentioned above are satisified.

These results are easily generalized to the case in which the branch cut is not a straight line on the real axis but a curve in the complex plane.

F. Fourier Integrals and Fourier Series

In this section we shall discuss the Fourier integral and the Fourier series, which are necessary tools for solving many physical problems. We shall encounter some of these problems in Chapters 4 and 5.

Let $F(x)$ be a function defined for all values of x from $-\infty$ to ∞. We define the Fourier transform of $F(x)$ as

$$\tilde{F}(k) \equiv \int_{-\infty}^{\infty} e^{-ikx} F(x)dx. \tag{2.53}$$

The Fourier transform $\tilde{F}(k)$ is uniquely determined once $F(x)$ is given, provided that the integral above is convergent.

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The Fourier integral theorem says that the converse is also true, that once $\tilde{F}(k)$ is given, one is able to determine $F(x)$ from $\tilde{F}(k)$. Indeed, the inversion formula of Fourier transform is

$$F(x) = \int_{-\infty}^{\infty} e^{ikx} \tilde{F}(k) \frac{dk}{2\pi} \; (-\infty < x < \infty), \tag{2.54}$$

which has a striking resemblance to (2.53).

With some partial differential equations, it is easier to find $\tilde{F}(k)$ than to find $F(x)$. One of the best ways to solve such equations is to find $\tilde{F}(k)$ first and then use Fourier's inversion theorem to determine $F(x)$.

To prove Fourier's inversion formula, we define

$$I_\lambda(x) \equiv \int_{-\lambda}^{\lambda} e^{ikx} \tilde{F}(k) \frac{dk}{2\pi}. \tag{2.55}$$

Then (2.54) is exactly

$$F(x) = \lim_{\lambda \to \infty} I_\lambda(x).$$

We substitute (2.53) into (2.55) and get

$$I_\lambda(x) \equiv \int_{-\lambda}^{\lambda} e^{ikx} \frac{dk}{2\pi} \left[\int_{-\infty}^{\infty} e^{-ikx'} F(x') dx' \right].$$

In the double integral above, we are supposed to integrate over x' first before we integrate over k. We shall reverse the order of the integration, integrating over k before integrating over x'. There are conditions on $F(x)$ under which this change of the order of integration is justified, but we will not elaborate on it here.

It is easy to carry out the integration over k and get

$$\int_{-\lambda}^{\lambda} e^{ik(x-x')} \frac{dk}{2\pi} = \frac{\sin[\lambda(x'-x)]}{\pi(x'-x)}. \tag{2.56}$$

Thus $I_\lambda(x)$ is given by

$$I_\lambda(x) \equiv \int_{-\infty}^{\infty} \frac{\sin[\lambda(x'-x)]}{\pi(x'-x)} F(x') dx'. \tag{2.57}$$

For the sake of getting the point across quickly, we shall first give a heuristic proof of the Fourier integral theorem without regard to rigor. We put

$$(x'-x)\lambda \equiv X; \tag{2.58}$$

then (2.57) becomes

$$I_\lambda(x) \equiv \int_{-\infty}^{\infty} \frac{\sin X}{\pi X} F\left(x + \frac{X}{\lambda}\right) dX. \tag{2.59}$$

Since

$$\lim_{\lambda\to\infty} F\left(x + \frac{X}{\lambda}\right) = F(x), \tag{2.60}$$

we have

$$\lim_{\lambda\to\infty} I_\lambda(x) = F(x) \int_{-\infty}^{\infty} \frac{\sin X}{\pi X} dX = F(x), \tag{2.61}$$

where we have made use of (2.38). This is the Fourier integral theorem (2.54).

As I have warned the reader, there is a lack of mathematical rigor with the arguments given above. This is because (2.61) is obtained after we replace $F(x + X/\lambda)$ in (2.59) with $F(x)$. Such a replacement is valid only if X/λ can be regarded as very small when λ is very large. But X, the integration variable of (2.59), ranges from $-\infty$ to ∞, and can surely be larger than λ. A rigorous proof must therefore include the argument that, as λ is very large, the region of integration that contributes to the integral (2.59) is restricted to

$$|X| \ll \lambda,$$

or, by (2.58),

$$|x' - x| \ll 1.$$

To prove this, we return to the integral in (2.57). The integration variable of this integral is x'. At $x' = x$, we have, by l'Hopital's rule,

$$\lim_{x'\to x} \frac{\sin[\lambda(x'-x)]}{\pi(x'-x)} = \frac{\lambda}{\pi}.$$

Therefore, the integrand of (2.57) at $x' = x$ is equal to $\lambda F(x)/\pi$, a very large number when λ is very large. It is clear that the integrand of (2.57) continues to be of the order of $\lambda F(x)$ in a sufficiently small interval around the point $x' = x$. To be specific, if $(x' - x)$ is as small as $1/\lambda$, then $(x' - x)^{-1}$ is as large as λ. Furthermore, since $(x' - x)$ is small, $F(x')$ is approximately $F(x)$. Thus the integrand of (2.57) is of the order of $\lambda F(x)$ in the small interval in which $(x' - x)$ is of the order of $1/\lambda$. The value of the integral over this small interval is of the order of the magnitude of the integrand times the width of the interval, or

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$$[\lambda F(x)] \cdot (1/\lambda) = F(x).$$

This order of magnitude estimate is in agreement with (2.61).

But, as we have mentioned, we need to prove that we may replace $F(x')$ by $F(x)$ for the entire integral of (2.57). The problem is that when x' is considerably different from x, the integrand of (2.57) does not vanish as $\lambda \to \infty$. For such values of x', it is not justified to replace $F(x')$ by $F(x)$.

We note that when λ is very large, the factor $\sin[\lambda(x'-x)]$ in the integrand is a rapidly oscillatory function of x'. As we shall discuss in more detail in Chapter 9, integrating a rapidly oscillatory function over an interval gives a very small number. For example,

$$\int_0^1 \cos(\lambda x')dx' = \frac{\sin\lambda}{\lambda},$$

which vanishes as λ goes to infinity. Therefore, with the exception of the small interval around $x' = x$ we just discussed, the contributions to the integral of (2.57) from any other region are very small. And as $\lambda \to \infty$, the contributions from any region other than an infinitesimal interval around $x' = x$ vanish. Thus, in the limit $\lambda \to \infty$, the contributions to the integral (2.57) come exclusively from an infinitesimal neighborhood of the point $x' = x$. For this reason, we may replace $F(x')$ in the integrand of (2.57) by $F(x)$ and get

$$\lim_{\lambda\to\infty} I_\lambda(x) = F(x) \lim_{\lambda\to\infty} \int_{-\infty}^{\infty} \frac{\sin[\lambda(x'-x)]}{\pi(x'-x)} dx'.$$

By (2.38), we get precisely (2.61), and the inversion formula for Fourier transform has been proven.

As a side remark,

$$\lim_{\lambda\to\infty} \frac{\sin[\lambda(x'-x)]}{\pi(x'-x)} \equiv \delta(x'-x) \tag{2.62}$$

is known to be the Dirac delta function. The Dirac delta function is very useful with many problems in science and engineering, and I shall say a few words about its properties here. First of all, by (2.38) we have

$$\int_{-\infty}^{\infty} \delta(x'-x)dx' = 1.$$

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Furthermore, since there is no contribution from any region outside an infinitesimal neighborhood of $x' = x$, we have, for any function $g(x')$,

$$\int_{-\infty}^{\infty} \delta(x'-x)g(x')dx' = g(x)\int_{-\infty}^{\infty} \delta(x'-x)dx' = g(x).$$

In particular,

$$\int_{-\infty}^{\infty} \delta(x')e^{-ikx'}dx' = 1,$$

which says that the Fourier transform of $\delta(x)$ is unity. By the inversion formula of Fourier transform, we find

$$\delta(x) = \int_{-\infty}^{\infty} e^{ikx}\frac{dk}{2\pi}, \tag{2.63}$$

which is the integral representation for the Dirac delta function. (The Dirac delta function will be discussed more fully in Chapter 4.)

Problem for the Reader

Find the Fourier transform of $F(x) = \dfrac{1}{4+x^2}$, $-\infty < x < \infty$.

Solution

We have

$$\tilde{F}(k) = \int_{-\infty}^{\infty} \frac{e^{-ikx}}{4+x^2}dx.$$

If $k > 0$, we close the contour downstairs and get

$$\tilde{F}(k) = -2\pi i\frac{e^{-2k}}{-4i} = \frac{\pi e^{-2k}}{2}.$$

If $k < 0$, we close the contour upstairs and get

$$\tilde{F}(k) = 2\pi i\frac{e^{2k}}{4i} = \frac{\pi e^{2k}}{2}.$$

Thus

$$\tilde{F}(k) = \frac{\pi e^{-2|k|}}{2}.$$

In the proof of the Fourier integral theorem, we have implicitly assumed that $F(x)$ is continuous. Consider now the case when $F(x)$ is discontinuous at x_0. We shall denote $F(x_0^+)$ as the value of $F(x)$ as x approaches x_0 from the right, and $F(x_0^-)$ as the value of $F(x)$ as x approaches x_0 from the left. We write (2.59) as

$$I_\lambda(x_0) \equiv \int_{-\infty}^{0} \frac{\sin X}{\pi X} F\left(x_0 + \frac{X}{\lambda}\right) dX + \int_{0}^{\infty} \frac{\sin X}{\pi X} F\left(x_0 + \frac{X}{\lambda}\right) dX. \quad (2.64)$$

In the limit $\lambda \to \infty$, $F(x_0 + \frac{X}{\lambda})$ approaches $F(x_0^-)$ if X is negative, and approaches $F(x_0^+)$ if X is positive. Therefore, we have

$$\lim_{\lambda \to \infty} I_\lambda(x_0) = \frac{F(x_0^-) + F(x_0^+)}{2},$$

which is, explicitly,

$$\int_{-\infty}^{\infty} e^{ikx_0} \tilde{F}(k) \frac{dk}{2\pi} = \frac{F(x_0^-) + F(x_0^+)}{2}. \quad (2.65)$$

This says that the Fourier integral of a function $F(x)$ is equal to the mean of the two values of $F(x)$ at x_0, at which $F(x)$ is discontinuous.

Problem for the Reader

Let

$$F(x) = e^{-x}, \ x > 0,$$
$$= 0, \quad x < 0.$$

This function is continuous at $x = 0$. Find $\tilde{F}(k)$ and verify if (2.65) is valid.

Solution

It is easy to find that

$$\tilde{F}(k) = \int_0^\infty e^{-ikx} e^{-x} dx = \frac{1}{1 + ik}.$$

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Verifying (2.65) is less trivial. The left side of (2.65) with $x_0 = 0$ is

$$\int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{1}{1+ik},$$

which is divergent.

We note that by (2.55), the lower limit and the upper limit of the Fourier inversion integral are $-\lambda$ and λ, respectively, with the limit $\lambda \to \infty$ taken after the integration has been carried out. Thus the divergent integral above should be regarded as

$$\begin{aligned}\lim_{\lambda\to\infty} \int_{-\lambda}^{\lambda} \frac{dk}{2\pi} \frac{1}{1+ik} &= \lim_{\lambda\to\infty} \int_{-\lambda}^{\lambda} \frac{dk}{4\pi} \left(\frac{1}{1+ik} + \frac{1}{1-ik} \right) \\ &= \lim_{\lambda\to\infty} \int_{-\lambda}^{\lambda} \frac{dk}{2\pi} \frac{1}{1+k^2} = \frac{1}{2},\end{aligned}$$

a finite result which agrees with the right side of (2.65).

Next we write, in (2.53),

$$e^{-ikx} = \cos kx - i \sin kx.$$

Then (2.53) becomes

$$\tilde{F}(k) = A(k) - iB(k),$$

where

$$A(k) = \int_{-\infty}^{\infty} \cos kx F(x) dx, \; B(k) = \int_{-\infty}^{\infty} \sin kx F(x) dx. \tag{2.66}$$

We find from (2.66) that

$$A(k) = A(-k), \quad B(k) = -B(-k).$$

Replacing $\tilde{F}(k)$ in the inversion formula (2.54) by $A(k) - iB(k)$, we get, after making use of the facts that $A(k)$ is an even functon of k and $B(k)$ is an odd function of k,

$$F(x) = \int_{-\infty}^{\infty} \left[A(k) \cos kx + B(k) \sin kx \right] \frac{dk}{2\pi}. \tag{2.67}$$

If $F(x)$ is an even function of x, (2.66) shows that $B(k) = 0$. In this case, $F(x)$ has the integral representation

$$F(x) = \int_{-\infty}^{\infty} A(k) \cos kx \frac{dk}{2\pi}. \tag{2.68}$$

If $F(x)$ is an odd function of x, (2.66) shows that $A(k) = 0$. Therefore, $F(x)$ has the integral representation

$$F(x) = \int_{-\infty}^{\infty} B(k) \sin kx \frac{dk}{2\pi}. \tag{2.69}$$

We shall next derive the Fourier integral representations of a function $F(x)$, which is given only in the semi-infinite region $0 < x < \infty$.

We may, as a mathematical artifact, define the function $F(x)$ for x negative by

$$F(-x) = F(x).$$

The function $F(x)$ is now defined for all values of x, and is an even function of x. As we have shown, an even function of x has the Fourier cosine integral representation. Thus a function originally defined only in the semi-infinite region $0 < x < \infty$ has a Fourier cosine integral representation given by (2.68). By (2.66) and the fact that $F(x)$ is even, we may express the coefficient $A(k)$ as

$$A(k) = 2\int_{0}^{\infty} \cos kx F(x) dx. \tag{2.70}$$

In this expression for $A(k)$, the region of integration is the positive x axis, on which the value of $F(x)$ is originally given.

Alternately, we may also define $F(x)$ for negative values of x by

$$F(-x) = -F(x).$$

The function is now defined for all values of x, and is an odd function of x. An odd function of x has the Fourier sine integral representation. Thus a function $F(x)$ given only for $x > 0$ has the Fourier sine integral representation (2.69), where

$$B(k) = 2\int_{0}^{\infty} \sin kx F(x) dx. \tag{2.71}$$

Note that the Fourier sine integral (2.69) vanishes as we set $x = 0$. If $F(0^+)$ is not zero, the Fourier sine integral at $x = 0$ is not equal to $F(0^+)$. Instead, the Fourier sine integral at $x = 0$ is equal to half of $F(0^+) + F(0^-)$, which is zero as $F(x)$ is by construction an odd function of x.

Next we turn to representations of functions given in a finite region. As we will see, such a function is represented by an infinite sum, not an integral. Let $f(\theta)$ be a function given in the interval $[-\pi, \pi]$, and let

$$a_n \equiv \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-in\theta} f(\theta) d\theta. \tag{2.72}$$

The coefficient a_n, known as the Fourier coefficient of $f(\theta)$, is determined once $f(\theta)$ is given. We shall prove that

$$f(\theta) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta}. \tag{2.73}$$

The series in (2.73) is known as the Fourier series of $f(\theta)$. It shows that the function $f(\theta)$ is determined once the Fourier coefficient a_n is given.

Since $e^{in\theta}$is a periodic function of θ with the period 2π, so is the Fourier series in (2.73). Therefore, if only implicitly so, the function $f(\theta)$ outside the interval $[-\pi, \pi]$ is assumed to be a periodic function of θ satisfying

$$f(\theta + 2\pi) = f(\theta). \tag{2.74}$$

Before we prove the inversion formula (2.73), we first say a few words about why such a formula is expected. We multiply (2.73) by $e^{-im\theta}/(2\pi)$ and integrate the equation from $-\pi$ to π. We get

$$\frac{1}{2\pi}\int_{0}^{2\pi} e^{-im\theta} f(\theta) d\theta = \frac{1}{2\pi}\int_{0}^{2\pi} e^{-im\theta} \sum_{n=-\infty}^{\infty} a_n e^{in\theta} d\theta. \tag{2.75}$$

The integrand on the right side of (2.75) is an infinite series, and there are conditions under which it is justified to carry out the integration term by term. We shall assume such conditions are satisfied. Since

$$\begin{aligned} \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i(m-n)\theta} d\theta &= 1, \ (m = n), \\ &= 0, \ (m \neq n), \end{aligned} \tag{2.76}$$

the only term on the right side of (2.75) that does not vanish after the integration has been performed is the term with $m = n$. Thus the right side of (2.75) is equal to a_m and we get (2.72). We have therefore shown that, if the function $f(\theta)$ given in the region $-\pi < \theta < \pi$ has a Fourier series expansion, its Fourier coefficient must be the one given by (2.72).

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The Fourier coefficient a_n of $f(\theta)$ can be compared with the component A_n of a vector $\overrightarrow{v}$ in space:

$$\overrightarrow{v} = A_1\overrightarrow{e_1} + A_2\overrightarrow{e_2} + A_3\overrightarrow{e_3}, \tag{2.77}$$

where $\overrightarrow{e_1}$, $\overrightarrow{e_2}$, and $\overrightarrow{e_3}$ are the unit vectors in the directions of the x, y,and z axes, respectively. Since the unit vectors $\overrightarrow{e_1}$, $\overrightarrow{e_2}$, and $\overrightarrow{e_3}$ are mutually orthogonal, the scalar product between them is given by

$$\begin{aligned} \overrightarrow{e_m} \cdot \overrightarrow{e_n} &= 1,\ m = n, \\ &= 0,\ m \neq n. \end{aligned} \tag{2.78}$$

From (2.77) and (2.78) we have

$$A_n = \overrightarrow{e_n} \cdot \overrightarrow{v}. \tag{2.79}$$

We note that (2.77) is the analogue of (2.73), with $\overrightarrow{e_n}$ the counterpart of $e^{in\theta}$. Also, (2.78) is the analogue of (2.76), and (2.79) is the analogue of (2.72). Note also that it is not possible to represent a vector $\overrightarrow{v}$ in the three-dimensional space by only two of its components, say A_1 and A_2, for

$$\overrightarrow{v} \neq A_1\overrightarrow{e_1} + A_2\overrightarrow{e_2}.$$

Rather, we need all three of the basis vectors $\overrightarrow{e_1}$, $\overrightarrow{e_2}$, and $\overrightarrow{e_3}$ to represent a three-dimensional vector $\overrightarrow{v}$. Such a set of basis vectors is said to be a complete set of basis vectors in the three-dimensional space. Returning to the issue of the Fourier series, (2.73) says that the set of functions $e^{in\theta}, n = 0, \pm1,\pm2, \cdots$ is complete. In other words, it is possible to use this set of functions to represent a function $f(\theta)$ in the form (2.73) for $-\pi \leq \theta \leq \pi$.

To prove (2.73), we define

$$S_N(\theta) \equiv \sum_{n=-N}^{N} a_n e^{in\theta}. \tag{2.80}$$

The remaining task is to show that

$$f(\theta) = \lim_{N\to\infty} S_N(\theta).$$

Substituting (2.72) into (2.80), we get

$$S_N(\theta) \equiv \sum_{n=-N}^{N} \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{in(\theta-\theta')} f(\theta') d\theta'. \tag{2.81}$$

Next we carry out the summation over n in (2.81). Since

$$\sum_{n=-N}^{N} \omega^n = \omega^{-N}(1+\omega+\cdots+\omega^{2N}) = \omega^{-N}\frac{1-\omega^{2N+1}}{1-\omega}$$
$$= \frac{\omega^{-N}-\omega^{N+1}}{1-\omega},$$

we have, by multiplying both the numerator and the denominator of the expression above by $\omega^{-1/2}$,

$$\sum_{n=-N}^{N} \omega^n = \frac{\omega^{-(N+1/2)}-\omega^{N+1/2}}{\omega^{-1/2}-\omega^{1/2}}.$$

Identifying $e^{i\theta}$ with ω, we get

$$\sum_{n=-N}^{N} e^{in\theta} = \frac{e^{-i(N+1/2)\theta}-e^{i(N+1/2)\theta}}{e^{-i\theta/2}-e^{i\theta/2}} = \frac{\sin(N+1/2)\theta}{\sin(\theta/2)}. \tag{2.82}$$

Thus (2.81) is

$$S_N(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\sin\left[(N+1/2)(\theta'-\theta)\right]}{\sin\left[(\theta'-\theta)/2\right]} f(\theta')d\theta'. \tag{2.83}$$

We shall consider S_n when N is very large. At a point $\theta' \neq \theta$, the integrand above is a rapidly varying function of θ'. And at $\theta' = \theta$, the integrand is as large as $(2N+1)f(\theta)$. As we have just explained, this means that the contributions to the integral come from a small neighborhood of $\theta' = \theta$. We may therefore make the approximation $\sin\left[(\theta'-\theta)/2\right] \approx (\theta'-\theta)/2$, and (2.83) becomes

$$S_N(\theta) \approx \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{\sin\left[(N+1/2)(\theta'-\theta)\right]}{(\theta'-\theta)} f(\theta')d\theta'. \tag{2.84}$$

By (2.62) and (2.84), we get

$$\lim_{N\to\infty} S_n(\theta) = f(\theta), \tag{2.85}$$

which is (2.73).

In the argument above, we have implicitly assumed that $f(\theta)$ is continuous. Let $f(\theta)$ be discontinuous at θ_0, with $f(\theta_0^+)$ the value of $f(\theta)$ as θ approaches θ_0 from the right, and $f(\theta_0^-)$ the value of $f(\theta)$ as θ approaches θ_0 from the left. It is then straightforward to prove that

$$\lim_{N\to\infty} S_N = \frac{1}{2}\left[f(\theta_0^+) + f(\theta_0^-)\right]. \tag{2.86}$$

Thus at a point of discontinuity, the Fourier series of a function is equal to the mean of the left and right limiting values of the function.

Problem for the Reader

Find the Fourier coefficient a_n for the function

$$f(\theta) = 1, \ (0 < \theta < \pi) \\ = -1, (-\pi < \theta < 0). \tag{2.87}$$

What are the values of the series at $\theta = 0$, $\pi/2$, and π? Can you explain why the series takes such values at these points?

Solution

We have

$$a_n = -\frac{1}{2\pi}\int_{-\pi}^{0} e^{-in\theta}d\theta + \frac{1}{2\pi}\int_{0}^{\pi} e^{-in\theta}d\theta = \frac{1-(-1)^n}{\pi in},$$

or

$$a_n = \frac{2}{\pi in}, (n \text{ odd}) \\ = 0, (n \text{ even}).$$

By (2.73), the Fourier series for $f(\theta)$ is

$$\frac{2}{\pi i}\left[\frac{e^{i\theta}}{1} - \frac{e^{-i\theta}}{1} + \frac{e^{3i\theta}}{3} - \frac{e^{3i\theta}}{3} + \cdots\right],$$

or

$$\frac{4}{\pi}\left[\frac{\sin\theta}{1} + \frac{\sin 3\theta}{3} + \cdots\right]. \tag{2.88}$$

At the point $\theta = \pi/2$, $f(\theta)$ is continuous. Therefore, the value of the Fourier series at $\theta = \pi/2$ is equal to the value of $f(\theta)$ at $\theta = \pi/2$. This gives us the identity

$$\frac{4}{\pi}\left[1 - \frac{1}{3} + \frac{1}{5} - \cdots\right] = 1.$$

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The Fourier series (2.88) vanishes at $\theta = 0$. To understand why, we note that the function $f(\theta)$ is discontinuous at $\theta = 0$, with

$$f(0^-) = -1, f(0^+) = 1.$$

By (2.86), the Fourier series of $f(\theta)$ must vanish at $\theta = 0$. The Fourier series (2.88) also vanishes at $\theta = \pi$. To understand why, we note that $f(\theta)$ is equal to unity for $0 < \theta < \pi$. Thus we have

$$f(\pi^-) = 1.$$

The value $f(\pi^+)$ is not explicitly given. But the function $f(\theta)$ is regarded as periodic with the period 2π. By (2.74), we have

$$f(\theta) = -1, (\pi < \theta < 2\pi),$$

and hence

$$f(\pi^+) = -1.$$

By (2.86), the Fourier series of $f(\theta)$ must vanish at $\theta = \pi$.

Next we derive another series known as the complete Fourier series for a function $f(\theta)$. We substitute

$$e^{in\theta} = \cos n\theta + i \sin n\theta$$

into (2.73) and get

$$f(\theta) = a_0 + \sum_{n=1}^{\infty} \left[A_n \cos n\theta + B_n \sin n\theta \right], \tag{2.89}$$

where

$$A_n = a_n + a_{-n}, B_n = i(a_n - a_{-n}), (n > 0). \tag{2.90}$$

Equation (2.89) is the complete Fourier series for $f(\theta)$. From (2.72) and (2.90), we have

$$A_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \cos n\theta d\theta, B_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \sin n\theta d\theta. \tag{2.91}$$

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If $f(\theta)$ is an even function of θ, we find from (2.91) that

$$B_n = 0,$$

and the complete Fourier series (2.89) is reduced to

$$f(\theta) = a_0 + \sum_{n=1}^{\infty} A_n \cos n\theta, \tag{2.92}$$

which is called a Fourier cosine series. The coefficients A_n and a_0 are given by

$$A_n = \frac{2}{\pi}\int_0^{\pi} f(\theta)\cos n\theta d\theta, \tag{2.93}$$

and

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(\theta) d\theta, \tag{2.94}$$

which are integrals over positive values of θ only.

Similarly, if the function $f(\theta)$ is an odd function of θ, we have from (2.72) and (2.91) that

$$a_0 = A_n = 0.$$

Therefore, the complete Fourier series (2.89) is reduced to the Fourier sine series

$$f(\theta) = \sum_{n=1}^{\infty} B_n \sin n\theta, \tag{2.95}$$

where

$$B_n = \frac{2}{\pi}\int_0^{\pi} f(\theta)\sin n\theta d\theta. \tag{2.96}$$

An example of an odd function of θ is the function of (2.87). As we have found, the series (2.88) for this function is indeed a Fourier sine series.

If a function $f(\theta)$ is given only in the interval $0 < \theta < \pi$, we may, as a mathematical artifact, define this function for θ negative to be

$$f(-\theta) = -f(\theta).$$

In this way, we have given meaning to the function $f(\theta)$ in the region $-\pi < \theta < 0$; the function so defined is an odd function of θ. As we have seen, an odd function can be represented by the Fourier sine series (2.95). This means that a function $f(\theta)$ given for $0 < \theta < \pi$ can be represented by the Fourier sine series (2.95) with the coefficient B_n given by (2.96).

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We may also extend the domain of a function $f(\theta)$ given for $0 < \theta < \pi$ by choosing it to satisfy

$$f(-\theta) = f(\theta).$$

The function $f(\theta)$ is then an even function of θ with the domain extended to $[-\pi, \pi]$. This function can be represented by the Fourier cosine series (2.92) with the coefficients A_n and a_0 given by (2.93) and (2.94).

We may also extend the results above to a function of x, which has the domain of $[-L, L]$, where L does not have to be equal to π. We define

$$\theta \equiv \frac{\pi}{L} x; \tag{2.97}$$

then the domain of θ is $[-\pi, \pi]$. By replacing θ in (2.73) with $\pi x/L$, we find that

$$F(x) = \sum_{n=-\infty}^{\infty} a_n e^{in\pi x/L}, (-L < x < L). \tag{2.98}$$

Making the same replacement in (2.72), we find that

$$a_n = \frac{1}{2L} \int_{-L}^{L} e^{-in\pi x/L} F(x) dx. \tag{2.99}$$

If the domain of a function $F(x)$ is $0 < x < L$, we may express this function as a Fourier cosine series

$$F(x) = a_0 + \sum_{n=1}^{\infty} A_n \cos(n\pi x/L), (0 < x < L). \tag{2.100}$$

The coefficients in the series above are given by

$$a_0 = \frac{1}{L} \int_0^L F(x) dx \tag{2.101}$$

and

$$A_n = \frac{2}{L} \int_0^L F(x) \cos(n\pi x/L) dx. \tag{2.102}$$

This function can also be expressed by the Fourier sine series

$$F(x) = \sum_{n=1}^{\infty} B_n \sin(n\pi x/L), (0 < x < L) \tag{2.103}$$

where

$$B_n = \frac{2}{L} \int_0^L F(x) \sin(nx/L) dx. \tag{2.104}$$

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We close this section with two remarks:

a. Let us compare the Fourier series $\sum_{n=-\infty}^{\infty} a_n e^{in\theta}$ with the Maclaurin series, with which the readers are perhaps more familiar. The convergence of the Maclaurin series depends on the magnitude of z. Specifically, the series converges faster if the magnitude of z is smaller. The situation is different for a Fourier series. This is because the magnitude of $e^{in\theta}$ is unity for all real values of θ. Therefore, there is no a priori reason why a Fourier series should always converge faster for $\theta = 0$ than for $\theta = \pi/2$, say. As a matter of fact, if we put $z \equiv e^{i\theta}$, the Fourier series (2.73) becomes $\sum_{n=-\infty}^{\infty} a_n z^n$, which is a Laurent series. Since the magnitude of $z = e^{i\theta}$ is equal to unity for all real values of θ, there is no a priori reason why the convergence of the series favors any particular value of θ.

In Chapters 4 and 5, we shall show how to express a solution of a partial differential equation by a Fourier series. We shall find that each term of the series represents a mode of the solution of the equation. The sum over the first few modes often approximates the solution equally well for all values of the independent variable.

b. Next we discuss an interesting phenomenon of the Fourier series known as the Gibbs phenomenon. As we know, $S_N(\theta)$ of (2.80) is the sum of a finite number of terms each of which is a continuous function of θ. Thus $S_N(\theta)$ is a continuous function of θ. Let $f(\theta)$ be discontinuous at θ_0. When N is very large, we expect S_N to approximate $f(\theta)$ very well. But how does the continuous function $S_N(\theta)$ approximate the discontinuous function $f(\theta)$ in the neighborhood of θ_0?

To be more specific, take the example of the function defined in (2.87). When N is large, we expext that $S_N(\theta)$ is approximately 1 when $0 < \theta < \pi$, and is approximately -1 when $-\pi < \theta < 0$. How does this continuous function $S_N(\theta)$ transit from roughly -1 to roughly 1 within a very short interval of θ around $\theta = 0$?

One may expect that the function $S_N(\theta)$ increases monotonically from approximately -1 to approximately 1 over a small neighborhood of the origin. But as it turns out, this is not exactly the case.

If $f(\theta)$ is the function of (2.87), we have from (2.84) that

$$
\begin{aligned}
S_N(\theta) \approx &-\frac{1}{\pi}\int_{-\pi}^{0} \frac{\sin\left[(N+1/2)(\theta'-\theta)\right]}{(\theta'-\theta)} d\theta' \\
&+ \frac{1}{\pi}\int_{0}^{\pi} \frac{\sin\left[(N+1/2)(\theta'-\theta)\right]}{(\theta'-\theta)} d\theta'.
\end{aligned}
$$

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Let

$$(N+1/2)(\theta'-\theta)=t;$$

then

$$S_N(\theta)\approx -\frac{1}{\pi}\int_{-(N+1/2)(\pi+\theta)}^{-(N+1/2)\theta}\frac{\sin t}{t}dt+\frac{1}{\pi}\int_{-(N+1/2)\theta}^{(N+1/2)(\pi-\theta)}\frac{\sin t}{t}dt.$$

In the limit of N approaching ∞, $(N+1/2)(\pi+\theta)$ and $(N+1/2)(\pi-\theta)$ both approach ∞, provided that θ is not near the endpoint $-\pi$ or the endpoint π. Thus we have

$$S_N(\theta)\approx -\frac{1}{\pi}\int_{-\infty}^{-(N+1/2)\theta}\frac{\sin t}{t}dt+\frac{1}{\pi}\int_{-(N+1/2)\theta}^{\infty}\frac{\sin t}{t}dt. \tag{2.105}$$

Since its integrand is an even function of t, the first integral in (2.105) is equal to

$$-\frac{1}{\pi}\int_{(N+1/2)\theta}^{\infty}\frac{\sin t}{t}dt.$$

The two integrals in (2.105) can be combined to give

$$S_N(\theta)\approx \frac{1}{\pi}\int_{-\Theta}^{\Theta}\frac{\sin t}{t}dt=\frac{2}{\pi}\int_{0}^{\Theta}\frac{\sin t}{t}dt, \tag{2.106}$$

where

$$\Theta=(N+1/2)\theta. \tag{2.107}$$

The integral of (2.106) is an odd function of Θ. Thus it suffices to discuss the behavior of this integral for nonnegative values of Θ only.

First, we consider $S_N(\theta)$ at $\theta=0$. We have from (2.106) that

$$\lim_{N\to\infty}S_N(0)=0,$$

which agrees with the average value of $f(0^+)$ and $f(0^-)$.

Next we consider $S_N(\theta)$ when θ takes a fixed and nonzero value. As $N\to\infty$, Θ goes to infinity. Thus we have

$$\lim_{N\to\infty} S_N(\theta) \approx \frac{2}{\pi}\int_0^\infty \frac{\sin t}{t}dt = 1,$$

which agrees with the value of $f(\theta)$ for $0 < \theta < \pi$.

Thus, when N is very large, the value of $S_N(\theta)$ changes from zero when θ is equal to zero to approximately unity when θ is equal to a fixed value, as expected.

But what are the values of $S_N(\theta)$ when the value of θ is in between? Let Θ, instead of θ, take a fixed, nonzero and finite value. This means that θ is equal to a fixed, nonzero and finite value divided by $(N + 1/2)$.

When Θ is finite, we have by (2.106) that $S_N(\theta)$ is approximately a function of Θ. Now $\sin t$ is positive for $0 < t < \pi$, is negative for $\pi < t < 2\pi$, and keeps changing its sign after a period of π. Thus the integrand $\sin t/t$ is positive for $0 < t < \pi$, is negative for $\pi < t < 2\pi$, and keeps changing its sign after a period of π. Since $1/t$ monotonically decreases as t increases, the maximum value of the integral

$$\frac{2}{\pi}\int_0^\Theta \frac{\sin t}{t}dt$$

is reached at $\Theta = \pi$, as the interval $0 < t < \pi$ is the maximum interval possible for $\sin t/t$ to be positive throughout. The value of $S_N(\theta)$ at $\Theta = \pi$ is

$$S_N\left(\frac{\pi}{N+1/2}\right) = \frac{2}{\pi}\int_0^\pi \frac{\sin t}{t}dt.$$

Numerically, this value is approximately 1.179, almost eighteen percent higher than the value of unity, the value of $f(\theta)$ for θ positive. Note that the value of θ at this point is $\pi/(N + 1/2)$, which goes to zero as $N \to \infty$. This shows that the value of the Fourier partial sum $S_N(\theta)$ does not change monotonically from 0 to 1. Rather, it starts at the value of zero at $\theta = 0$ and overshoots its target value by almost eighteen percent at $\theta = \pi/(N + 1/2)$.

Neither does $S_N(\theta)$ move down from its peak value to its target value monotonically. As Θ becomes larger than π, $S_N(\theta)$ decreases as the integrand of (2.106) becomes negative in the region of integration $t > \pi$. The function $S_N(\theta)$ reaches a minimum at $\theta = 2\pi/(N + 1/2)$ and then starts increasing again. In fact, it oscillates around the value of unity many times before it approaches the asymptotic value of unity. If we plot the function $S_N(\theta)$ as a function of θ, the distances between peaks and valleys shrink to zero as N increases to infinity, while the heights of the peaks and valleys stay constant.

This oscillatory behavior of $S_n(\theta)$ is called the Gibbs phenomenon.

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G. The Laplace Transform

Let a function $f(x)$ be given for $x \geq 0$. The Laplace transform for such a function is defined to be

$$L(s) = \int_0^\infty e^{-sx} f(x) dx, \tag{2.108}$$

provided that the integral converges. In the above, s is independent of x.

Problem for the Reader

Find the Laplace transforms of the following functions:

a. $x^n/n!$, $n \geq 0$,
b. e^{-ax},
c. $\cos kx$ and $\sin kx$,
d. e^{x^2}.

What is the analytic property of each of these Laplace transforms as a function of the complex variable s?

Solution

a. $x^n/n!$

It is known that

$$\int_0^\infty e^{-t} t^n dt = n!.$$

Therefore,

$$\int_0^\infty dx e^{-sx} x^n/n! = \frac{1}{s^{n+1}}. \tag{2.109}$$

The integral above is convergent only if $\mathrm{Re}\, s > 0$. If $\mathrm{Re}\, s \leq 0$, the Laplace integral above is divergent. Therefore, the Laplace transform of x^n is defined by the Laplace integral above only in the right half-plane $\mathrm{Re}\, s > 0$. We see that in this half-plane, s^{-n-1} is an analytic function of s.

When the integral above is not convergent, we call the right side of (2.109) the Laplace transform of $x^n/n!$. In other words, we simply define the Laplace transform of $x^n/n!$ in the left half-plane $\mathrm{Re}\, s \leq 0$ to be s^{-n-1}. The Laplace transform of $x^n/n!$ so defined is an analytic function of s with the exception of an n^{th}-order pole at $s = 0$. Thus we have analytically continued the Laplace transform of $x^n/n!$, defined by the Laplace integral in the right half-plane $\mathrm{Re}\, s > 0$, to the left half-plane $\mathrm{Re}\, s < 0$.

b. e^{-ax}

The Laplace transform of e^{-ax} is

$$\int_0^\infty dx e^{-sx} e^{-ax} = \frac{1}{s+a}. \tag{2.110}$$

The integral above is convergent only if $\mathrm{Re}(s+a) > 0$. We analytically continue this Laplace transform by defining it as $(s+a)^{-1}$ for all s. It is analytic everywhere with the exception of a simple pole at $s = a$.

c. $\cos kx$ and $\sin kx$

Replacing a in (2.110) by ik, we have

$$\int_0^\infty dx e^{-sx} e^{-ikx} = \frac{1}{s+ik}.$$

Restricting to real values of s and k, we obtain from the real part and the imaginary part of the equation above

$$\int_0^\infty dx e^{-sx} \cos kx = \frac{s}{s^2+k^2}, \tag{2.111}$$

and

$$\int_0^\infty dx e^{-sx} \sin kx = \frac{k}{s^2+k^2}. \tag{2.112}$$

The integrals in (2.111) and (2.112) converge if s is greater than zero. For complex values of s and k, we simply define the Laplace transforms of $\cos kx$ and $\sin kx$ to be the right sides of (2.111) and (2.112), respectively. The Laplace transforms so defined are analytic functions of s with simple poles at $s = \pm ik$.

d. e^{x^2}

As $x \to \infty$, the function e^{x^2} blows up so rapidly that $e^{-sx} e^{x^2}$ for any value of s always blows up as $x \to \infty$. As a result, the integral $\int_0^\infty e^{-sx} e^{x^2} dx$ is divergent no matter what s is. Thus the function e^{x^2} has no Laplace transform.

We note that the Laplace transforms of 1, e^{-ax}, and $\cos kx$ all approach $1/s$ as $|s| \to \infty$. As a general rule, the Laplace integral in (2.108) approaches $f(0)/s$ as $|s| \to \infty$ provided that $f(0)$ is finite and does not vanish. This can be shown by the methods we shall give in Chapter 8.

We observe that each $L(s)$ in the examples given is an analytic function of s in the region where its Laplace integral converges. This is true in a more general context. The integrand in (2.108) is an analytic function of s. Consequently, the integral in (2.108) is an analytic function of s in the region of s where it is convergent. Let

$$s = s_1 + is_2,$$

where s_1 and s_2 are the real part and the imaginary part of s, respectively. Now we have

$$\left|e^{-sx}\right| = e^{-s_1 x}.$$

As the value of x in the Laplace integral is always positive, $e^{-s_1 x}$ becomes smaller as s_1 becomes larger. Thus, if the integral of (2.108) converges at $s = \xi_0$, then the integral of (2.108) converges for all s satisfying $s_1 > \mathrm{Re}\,\xi_0$. Hence $L(s)$ is an analytic function of s in the right half-plane $\mathrm{Re}\,s > \mathrm{Re}\,\xi_0$. In particular, if $L(s)$ exists when s is purely imaginary, $L(s)$ is analytic in the right half-plane $s_1 \geq 0$. If such is the case, all of the possible singularities of $L(s)$ lie to the left of the imaginary axis.

The Laplace transform is a special case of the Fourier transform. Let

$$\begin{aligned} F(x) &= f(x),\ x > 0, \\ &= 0, \qquad x < 0, \end{aligned} \tag{2.113}$$

and let the Fourier integral of $F(x)$ converge. Then the Fourier integral of (2.53) is the same as the Laplace integral of (2.108) with s identified with ik. Thus we have

$$L(ik) = \tilde{F}(k). \tag{2.114}$$

This shows that if $\tilde{F}(k)$ exists when k is real, $L(s)$ exists when s is purely imaginary. If such is the case, (2.54) gives

$$f(x) = \int_{-\infty}^{\infty} e^{ikx} L(ik) \frac{dk}{2\pi}, \quad x > 0. \tag{2.115}$$

Replacing k by $-is$, we obtain

$$f(x) = \int_{-i\infty}^{i\infty} e^{sx} L(s) \frac{ds}{2\pi i}, \quad x > 0, \tag{2.116}$$

where the integration is over the imaginary s-axis. Equation (2.116) is an inversion formula that enables us to determine the values of $f(x)$ once the values of $L(s)$ on the imaginary axis of the s-plane are obtained.

We have implicitly assumed in the above that $f(x)$ is a continuous function of x. At the point x_0 where $f(x)$ is discontinuous, the left side of (2.116) should be replaced by

$$\frac{f(x_0^+) + f(x_0^-)}{2}.$$

In particular, if $f(0) \neq 0$, the function $F(x)$ of (2.113) is discontinuous at $x = 0$, and the inversion integral of (2.116) at $x = 0$ is equal to $f(0)/2$.

We may think of (2.116) as the contour integral

$$f(x) = \int_c e^{sx} L(s) \frac{ds}{2\pi i}, \quad x > 0, \tag{2.117}$$

where c is the imaginary axis of the complex s-plane. Since $L(s)$ is analytic in the right half-plane $\operatorname{Re} s \geq 0$ by assumption, all possible singularities of the integrand lie to the left of c.

The contour c does not have to be a straight line. We are allowed to deform c into another contour in conformity to the Cauchy integral theorem.

If we differentiate (2.108) with respect to s, we get

$$\frac{dL(s)}{ds} = -\int_0^\infty e^{-sx} x f(x)\, dx.$$

The integral above is convergent for $s_1 > \operatorname{Re} \xi_0$ if the Laplace integral converges at $s = \xi_0$. This is because $e^{-s_1 x}$ is smaller if s_1 is larger, as we have mentioned above. Therefore, the Laplace transform of $xf(x)$ is $-dL(s)/ds$.

On the other hand, if we differentiate (2.117) with respect to x, we get

$$\frac{df(x)}{dx} = \int_c e^{sx} sL(s) \frac{ds}{2\pi i}. \tag{2.118}$$

However, (2.118) is not always valid, as the integral in (2.118) is not always convergent. Indeed, if $f(0)$ is finite and is not equal to zero, $sL(s)$ approaches the constant $f(0)$ as $s \to \pm i\infty$, while e^{sx} is oscillatory and does not vanish as $s \to \pm i\infty$. Thus the integral in (2.118) does not converge if $f(0)$ does not vanish, and (2.118) is not meaningful. Therefore, $sL(s)$ is not always the Laplace transform of $df(x)/dx$.

Let us find out what the Laplace transform of df/dx is. We have

$$\int_0^\infty dx e^{-sx} df(x)/dx = -f(0) + \int_0^\infty dx\, s e^{-sx} f(x),$$

where we have performed an integration by parts. Thus the Laplace transform of df/dx is

$$-f(0) + sL(s), \tag{2.119}$$

which is equal to $sL(s)$ only if $f(0) = 0$.

Problem for the Reader

Express the Laplace transform of d^2f/dx^2 with $L(s)$.

Solution

After performing two integrations by parts successively, we find that

$$\int_0^\infty dx e^{-sx} d^2 f(x)/dx^2 = -f'(0) - sf(0) + s^2 L(s). \tag{2.120}$$

We mention that the integral in (2.117) is equal to zero if $x < 0$. This is because if $x < 0$, $e^{s_1 x}$ approaches zero as s_1 approaches $+\infty$. Therefore, the integrand in (2.117) vanishes at the infinity of the right-half s-plane and we may close the contour to the right. Since the integrand has no singularities in the right half-plane, the integral is zero by Cauchy's integral theorem. We may add that this is consistent with the definition of $F(x)$ for $x < 0$ as given by (2.113).

We have shown that the contour integral (2.117) is valid if $L(s)$ is analytic in the region $\text{Re}\, s \geq 0$. If this condition is not met, i.e., if $L(s)$ has singularities in the region $\text{Re}\, s \geq 0$, the only modification needed is to make c in (2.117) a vertical line that lies to the right of all of the singularities of $L(s)$. To see this, let us multiply $f(x)$ by the exponentially vanishing function e^{-ax} and call

$$g(x) = e^{-ax} f(x). \tag{2.121}$$

We shall assume that it is possible to choose a so that $g(x)$ vanishes sufficiently rapidly as $x \to \infty$ and the Laplace transform of $g(x)$, given by

$$G(s) = \int_0^\infty e^{-sx} g(x) dx, \tag{2.122}$$

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is analytic in the right half-plane $\mathrm{Re}\, s \geq 0$. By the inversion formula of Laplace transform, we get

$$g(x) = \int_{-i\infty}^{i\infty} e^{sx} G(s) \frac{ds}{2\pi i}. \tag{2.123}$$

Replacing $g(x)$ with $e^{-ax} f(x)$, we obtain from (2.122) that

$$G(s) = \int_0^{\infty} e^{-sx} e^{-ax} f(x) dx = L(s+a).$$

Since $G(s)$ is by assumption analytic in the region $\mathrm{Re}\, s \geq 0$, $L(s) = G(s-a)$ is analytic in the region $\mathrm{Re}\, s \geq a$.

Replacing $g(x)$ with $e^{-ax} f(x)$, we obtain from (2.123) that

$$f(x) = \int_{-i\infty}^{i\infty} e^{(s+a)x} L(s+a) \frac{ds}{2\pi i}.$$

This formula can be written as

$$f(x) = \int_c e^{sx} L(s) \frac{ds}{2\pi i}, \tag{2.124}$$

where c is the straight line $s = a + is_2$. Since $L(s)$ is analytic in the region $\mathrm{Re}\, s \geq a$, all of the possible singularities of $L(s)$ are at the left of the contour c.

Laplace transform is useful in solving differential equations with initial conditions. As an example, let us consider the solution of the differential equation

$$\frac{d^2 y}{dx^2} + y = 1, x > 0,$$

satisfying the initial conditions $y(0) = 1$, $y'(0) = 2$.

Problem for the Reader

Solve the initial-value problem above by Laplace transform.

Solution

By (2.120), the Laplace transform of the left side of the differential equation is

$$-2 - s + (s^2 + 1) L(s).$$

Note that the first two terms in the expression above are given by the initial conditions. The Laplace transform of the right side is equal to s^{-1}. Equating these two Laplace transforms gives

$$L(s) = \frac{s^{-1} + 2 + s}{1 + s^2}.$$

By (2.124), we have

$$y(x) = \int_c e^{sx} \frac{s^{-1} + 2 + s}{1 + s^2} \frac{ds}{2\pi i},$$

where c is a vertical line lying at the right of the imaginary s axis. For $x > 0$, e^{sx} vanishes as $s_1 \to -\infty$. Therefore, we may evaluate the integral above by closing the contour to the left. By Cauchy's residue theorem the integral is equal to the sum of the residues of the integrand at $s = 0, i$, and $-i$. Thus we get

$$y(x) = 1 + 2\sin x, \quad x \geq 0.$$

Note that the initial conditions have already been incorporated. Therefore, unlike under the method used in Chapter 1, there are no arbitrary constants in the solution obtained here, and we are spared the chore of determining the arbitrary constants with the use of the initial conditions. It is easy to verify that the initial conditions are indeed satisfied by this solution.

Homework Problems for This Chapter

Solutions to the Homework Problems can be found at www.lubanpress.com.

1. Prove that the limit of (2.5) is the same for any Δz if the Cauchy-Riemann equations are satisfied by the real part and the imaginary part of $f(z)$.

2. Prove that $\cos\frac{2\pi}{5} = \frac{\sqrt{5} - 1}{4}$.

Hint: Let $e^{\pi i/5} \equiv c + is$; then

$$(c + is)^5 = -1.$$

Equate the imaginary parts of the two sides of this equation.

3. A function that is analytic everywhere in the finite complex plane is called an entire function. Prove the Liouville theorem that an entire function $f(z)$ is a constant if it is bounded at infinity.

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Hint: Show that $f'(z) = 0$ with the formula

$$f'(z) = \frac{1}{2\pi i} \lim_{R\to\infty} \oint_{c_R} \frac{f(z')}{(z'-z)^2} dz',$$

where c_R is a circle with its center at the origin.

4. Evaluate the following integrals with contour integration:

a. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)(x-2i)(x-3i)(x-4i)}.$
Ans. $-\dfrac{i\pi}{60}.$

b. $\displaystyle\int_{-\infty}^{\infty} \frac{1-\cos 2x}{x^2} dx.$
Ans. $2\pi.$

c. $\displaystyle\int_{-\infty}^{\infty} \frac{\sin^3 x}{x^3} dx.$
Ans. $3\pi/4.$

d. $\displaystyle\int_{0}^{2\pi} \frac{1}{(a+b\cos\theta)^2} d\theta \ (a > b > 0).$
Ans. $2\pi a/(a^2-b^2)^{3/2}.$

e. $\displaystyle\int_{-\infty}^{\infty} \frac{x\sin x}{x^2-4\pi^2} dx.$
Ans. $\pi.$

5. Explain why the integral of (2.35) is not equal to the imaginary part of

$$\int_c \frac{e^{iz}}{z} dz$$

if c is the contour in Figure 2.7

6. Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{\sin x}{x+i} dx.$$

Ans. $\dfrac{\pi}{e}.$
Explain why it is not fruitful to evaluate the integral

$$J = \int_{-\infty}^{\infty} \frac{e^{ix}}{x+i} dx.$$

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7. Consider $f(z) = \log \dfrac{1+\sqrt{1+z^2}}{2}$.

a. Find all possible branch points of this function.
Ans. $0, \pm i, \infty$.

b. If we define $\sqrt{1+z^2}\ |_{z=0} = 1$, show that the origin is not a branch point of this function. Draw a set of branch cuts to make the function single-valued.

8. Show that the Taylor series (2.19) is convergent inside the circle with center at z_0 and with the radius equal to $|z_0 - z_1|$, where z_1 is the singularity of $f(z)$ nearest to z_0.

Hint: Estimate the magnitude of $f^{(n)}(z_0)$ with the use of the Cauchy integral formula.

9. Let $f(z)$ and $g(z)$ be analytic in a region R, and let z_0 be an interior point of R. If $f(z) = g(z)$ has at least one root in any neighborhood of z_0, no matter how small this neighborhood is, prove that $f(z) = g(z)$ in R.

Hint: Let $G(z) \equiv f(z) - g(z)$ and consider the Taylor series expansion of $G(z)$ around z_0. Show that unless this series vanishes identically, it cannot vanish at z if z is sufficiently close to z_0 but not equal to z_0.

10. Let $I_n = \displaystyle\int_0^\infty \frac{dx}{1+x^n}$.

a. Prove that $I_n = \dfrac{\pi}{n\sin(\pi/n)}$. What is I_n in the limit $n \to \infty$?

b. Show that as $n \to \infty$, the limit of the integral I_n is equal to unity integrated over $[0, 1]$.

11. Evaluate the following integrals making use of branch cuts:

a. $\displaystyle\int_0^\infty \frac{\ln^2 x}{1+x^2}dx$. Ans. $\pi^3/8$.

b. $\displaystyle\int_0^1 \frac{1}{1+x^2}\sqrt{\frac{x^3}{1-x}}dx$.
Hint: The integrand has a square-root branch cut from 0 to 1. One considers the sum of the following five contours: 1. The straight line joining $i\epsilon$ to $1 + i\epsilon$; 2. The straight line joining $1 - i\epsilon$ to $-i\epsilon$; 3. The right half-circle in the clockwise direction with center at 1 and with the radius ϵ; 4. The left half-circle in the clockwise direction with center at

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0 and with the radius ϵ; 5. The infinite circle in the counterclockwise direction. You may prove that as ϵ approaches zero, the contributions from contour 3 and contour 4 are zero, and the contribution from contour 2 is equal to that from contour 1. The contribution from contour 5 can be calculated. The sum of the contributions from these five contours is equal to, by Cauchy's residue theorem, $2\pi i$ times the sum of the residues of the integrand in the cut plane.
Ans. $\pi - \dfrac{\pi\cos(\pi/8)}{2^{1/4}}$.

c. $\displaystyle\int_0^\infty \frac{\ln x}{1+x^5}dx.$
Hint: Add to the contour of the positive real axis the ray joining zero to infinity with the argument $2\pi/5$. On this ray, $z^5 = r^5$, and $\ln z = \ln r + 2\pi/5$.

12. Let z_0 be an isolated singularity of $f(z)$, and let z be a point in the neighborhood of z_0. Show that

$$f(z) = \frac{1}{2\pi i}\int_{C_R}\frac{f(z')}{z'-z}dz' - \frac{1}{2\pi i}\int_{C_\epsilon}\frac{f(z')}{z'-z}dz',$$

where C_R and C_ϵ are counterclockwise circles, the centers of which are at z_0 and the radii of which are R and ϵ, respectively. Also, R is sufficiently large so that z is inside C_R, and ϵ is sufficiently small so that z is outside C_ϵ. Derive the Laurent series expansion of $f(z)$ from the equation above and discuss the region where the series is convergent.

13. Find the Fourier coefficient a_n for the following functions. What is the value of the Fourier series at $\theta = \pi$?

a. e^θ.
Ans. $a_n = \dfrac{(-1)^n}{2\pi}\dfrac{e^\pi - e^{-\pi}}{1-in}$. The value of the series at $\theta = \pi$ is $\dfrac{1}{2}(e^\pi + e^{-\pi})$.

b. $\dfrac{1}{a+b\cos\theta}$.
Ans. $a_n = (-1)^n\left(a-\sqrt{a^2-b^2}\right)^n / \left(b^n\sqrt{a^2-b^2}\right)$ $(n>0)$, $a_{-n} = a_n$. The value of the Fourier series at $\theta = \pi$ is $(a-b)^{-1}$.

14. Let u and v be functions of x and y. Prove that

$$(\overrightarrow{\nabla}u)\cdot(\overrightarrow{\nabla}v) \equiv u_x v_x + u_y v_y = 2(u_z v_{z^*} + u_{z^*} v_z).$$

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15. Show that if $f(z)$ is analytic everywhere in the complex plane with the exception of a branch cut from a to b on the real axis, does not blow up as fast as a pole at the endpoints a and b, and vanishes at infinity, then

$$f(z) = \int_a^b \frac{r(x')}{x'-z} dx',$$

with $2\pi i r(x)$ the discontinuity of $f(z)$ across the branch cut.

Hint: Consider the integral $\frac{1}{2\pi i}\int \frac{f(z')dz'}{z'-z}$ with the contour of integration a closed contour wrapping around the interval $[a, b]$ in the clockwise direction. Show that this integral is equal to $f(z)$ provided that z is outside this contour. Now make this contour to be infinitesimally close to the line from a to b on the real axis.

16. Find the Fourier transforms of the functions $e^{-|x|}$ and $(1+x^2)^{-2}$.

Ans. $2(1+k^2)^{-1}$ and $\pi(1+|k|)e^{-|k|}/2$.

17. The convolution of $F(x)$ and $G(x)$ is defined to be

$$H(x) = \int_{-\infty}^{\infty} F(x-x')G(x')\,dx'.$$

a. Calculate the Fourier transform of $H(x)$ and show that it is equal to $\tilde{F}(k)\tilde{G}(k)$, where $\tilde{F}(k)$ and $\tilde{G}(k)$ are the Fourier transforms of $F(x)$ and $G(x)$, respectively.

b. Use the result above and show that $\int_{-\infty}^{\infty} F(-x)G(x)\,dx$
$= \int_{-\infty}^{\infty} \tilde{F}(k)\tilde{G}(k)\frac{dk}{2\pi}$.

18. The convolution of $F_m(x)$, $m = 1, 2, \cdots, n$, is

$$H(x) = \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \cdots \int_{-\infty}^{\infty} dx_n\, \delta\left(x - \sum_{m=1}^{n} x_m\right) \prod_{n=1}^{n} F_m(x_m).$$

Show that the Fourier transform of $H(x)$ is $\Pi_{m=1}^{n}\tilde{F}_m(k)$, where $\tilde{F}_m(k)$ is the Fourier transform of $F_m(x)$.

19. Let $f(x)$ and $g(x)$ be defined for $x > 0$. The convolution of $f(x)$ and $g(x)$ is defined to be $\int_0^x f(x-x')g(x')\,dx'$. Show that the Laplace transform of the convolution of $f(x)$ and $g(x)$ is equal to the product of the Laplace transform of $f(x)$ and that of $g(x)$.

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20. Solve with the method of Laplace transform the initial-value problem

$$\frac{d^2y}{dx^2} + y = \sqrt{x},\ x > 0,$$

with the initial condition $y(0) = 1$, $y'(0) = 0$.

Hint: the Laplace inversion integral for $y(x)$ cannot be evaluated with Cauchy's residue theorem, as the integrand of this integral has a branch point. Make use of the result of Problem 19 and express $y(x)$ as a convolution of elementary functions.

Chapter 7

The WKB Approximation

The WKB method is a powerful tool to obtain solutions for many physical problems. It is generally applicable to problems of wave propagation in which the wave number of the wave is very high or, equivalently, the wavelength of the wave is very small. The WKB solutions are approximate solutions, but sometimes they are surprisingly accurate. In this chapter we'll discuss this method, which is applicable to linear equations only.

A. WKB in the Zeroth and the First Order

Consider the first-order linear differential equation

$$y' = ipy.$$

If p is a constant, the solution is simply

$$y = e^{ipx},$$

which describes a wave of propagation number p.

But if p is a function of x, the solution of this equation is

$$y(x) = \exp\left(i\int p(x)dx\right).$$

If we regard $y(x)$ as a wave, then the exponent of $y(x)$ is the phase of a wave with the wave number $p(x)$. We see that at the point x, the phase is equal to the waveshift accumulated over the path of propagation, not to x times the local wave number $p(x)$—a perfectly logical result.

We mention that a change of the lower limit of the integral above merely changes the integral by an additive constant, and hence y by a multiplicative constant. For a linear homogeneous equation, the freedom of multiplying a solution by a constant is always understood. Thus we shall leave the lower limit of integration unspecified.

Next we consider the second-order differential equation

$$y'' + p^2 y = 0. \tag{7.1}$$

If p is a constant, the two independent solutions of (7.1) are $\exp(\pm ipx)$, waves with wave number p travelling in opposite directions of the x axis.

If p is a function of x, it appears reasonable that the solutions are two waves with the phase $\pm \int p(x)dx$. Thus we may surmise that the independent solutions of (7.1) are

$$e^{\pm i \int p(x)dx}, \tag{7.2}$$

which are called the zeroth-order WKB solutions.

Let us see if these solutions satisfy (7.1). It is straightforward to show that

$$\left(\frac{d^2}{dx^2} + p^2\right) e^{\pm i \int p(x)dx} = \pm ip' e^{\pm i \int p(x)dx}. \tag{7.3}$$

Therefore, the WKB solutions (7.2) do not satisfy (7.1) unless $p' = 0$, or p is independent of x.

While we get a negative answer, (7.3) suggests that $\exp(\pm i \int p(x)dx)$ are good approximate solutions of (7.1) provided that $\pm ip'$ is negligible, or, more precisely, if

$$|p'| \ll p^2,$$

the right side of this inequality being a term inside the parentheses of (7.3). The inequality above can be written as

$$\left|\frac{d}{dx}\frac{1}{p(x)}\right| \ll 1. \tag{7.4}$$

In particular, this condition is satisfied if the wave number $p(x)$ is of the form

$$p(x) \equiv \lambda P(x), \tag{7.5}$$

where λ is a large constant, i.e.,

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$$\lambda \gg 1, \tag{7.6}$$

and $P(x)$ is of the order of unity. We have assumed that none of the constant parameters in $P(x)$ are large. Indeed, if $p(x)$ is given by (7.5), the inequality (7.4) is

$$\left| \frac{1}{\lambda} \frac{d}{dx} \frac{1}{P(x)} \right| \ll 1. \tag{7.7}$$

Clearly, (7.7) is satisfied if $\lambda \gg 1$, provided that x is not near a zero of $P(x)$.

As a side remark, while it is easy to accept that we may drop a term in the equation that is much smaller than a term in the equation being kept, we shall see in later chapters that this is not always a valid procedure. For example, the effects from a small term of the differential equation may add up, and as the solution evolves over a long interval of the independent variable, small perturbations may accumulate into a large correction. A justification of the WKB solutions will be given later.

We note that the phases of the solutions $\exp(\pm i \int p(x)dx)$ are functions of x, but the magnitudes of these approximate solutions are independent of x. Let us remember that, in Chapter 1, we have shown that if y_1 and y_2 are two independent solutions of (7.1), then the Wronskian $W(x) \equiv y_1 y_2' - y_1' y_2$ is independent of x. Now the Wronskian of $\exp(i \int p(x)dx)$ and $\exp(-i \int p(x)dx)$ is easily shown to be equal to $2ip(x)$, not a constant unless $p(x)$ is a constant. This suggests that these approximate solutions still leave something to be desired. Because the Wronskian of the approximate solutions miss by a factor of $p(x)$, let us try to fix it by adding an additional factor $1/\sqrt{p(x)}$ to each of the approximate solutions. The resulting approximate solutions are

$$y^{\pm}_{WKB}(x) = \frac{1}{\sqrt{p(x)}} e^{\pm i \int p(x)dx}, \tag{7.8}$$

which is said to include both the zeroth-order and the first-order terms of the WKB approximations.

The magnitude of these solutions varies with x like $1/\sqrt{p(x)}$. The Wronskian of $y^{\pm}_{WKB}(x)$ is now exactly a constant. (See homework problem 1.) It is therefore tempting to surmise that under the condition (7.4) or, equivalently, (7.7), $y^{\pm}_{WKB}(x)$ are even better approximations than

$$e^{\pm i \int p(x)dx}.$$

To see if this is true, we put

$$y \equiv e^{\pm i \int p(x)dx} v, \tag{7.9}$$

and substitute this expression of y into (7.1). We get

$$(D \pm ip)(D \pm ip)v + p^2 v = 0,$$

or

$$\left(\frac{d^2}{dx^2} \pm 2ip\frac{d}{dx} \pm ip' \right) v = 0.$$

We shall write the equation above as

$$v' + \frac{p'}{2p}v = \pm \frac{i}{2p}v''. \tag{7.10}$$

By (7.5), (7.10) can be written as

$$v' + \frac{P'}{2P}v = \pm \epsilon \frac{i}{2P}v'', \tag{7.11}$$

where

$$\epsilon \equiv 1/\lambda$$

is a small number. In the first-order approximation, we ignore the right side of (7.11) and we get

$$v' + \frac{P'}{2P}v \approx 0, \tag{7.12}$$

which gives

$$v(x) \approx \frac{1}{\sqrt{P(x)}}. \tag{7.13}$$

Thus (7.9) and (7.13) give, aside from an immaterial overall constant, the WKB solutions (7.8).

The more traditional way to derive the WKB solutions is given in homework problem 2 in this chapter.

We have mentioned that if $P(x)$ has a zero at x_0, the inequality (7.4) does not hold at x_0. To see how far away from x_0 it must be for the WKB approximation to hold, let $P(x)$ near x_0 be approximately given by

$$P(x) \approx a(x - x_0)^n, \;\; x \approx x_0. \tag{7.14}$$

Equation (7.7) requires

$$|x - x_0| \gg \left(\frac{n}{\lambda a}\right)^{1/(n+1)}. \tag{7.15}$$

Equation (7.15) tells us how far away from x_0 it must be for the WKB approximate solutions to be valid.

We mention that if $P(x)$ vanishes in the way given by (7.14), we say that $P(x)$ has an n^{th}-order zero at x_0. Not all zeroes of $P(x)$ are of finite order; an example of $P(x)$ having a zero of infinite order is given by homework problem 7.

As we have stated at the beginning of this chapter, the WKB approximation is useful for describing the propagation of waves with very small wavelengths. We shall now explain what this means more precisely. The wave number for the WKB solutions (7.2) is $p(x)$, and the corresponding wavelength $\Omega(x)$ is $2\pi/p(x)$. When λ of (7.5) is large, $p(x)$ is large and the wavelength $\Omega(x)$ is small. In the meantime, the inequality (7.4) is satisfied, justifying the WKB approximation.

While all of this is straightforward in mathematical terms, we deem it useful to clarify it further in physical terms. This is because the wavelength has the dimension of the distance, and the numerical value of a wavelength depends on the distance unit we choose. For example, 10^{-3} centimeters is exactly the same as 10^4 nanometers. While 10^{-3} is a small number, 10^4 is a large number. Is the wavelength of such a value small or large?

The fact is that it is not meaningful to classify a quantity as either small or large unless we compare it with another quantity of the same dimension. Let us first examine (7.4) in this light. Expressed in terms of $\Omega(x)$, (7.4) is

$$\left|\frac{d}{dx}\frac{\Omega(x)}{2\pi}\right| \ll 1.$$

As both $d\Omega(x)$ and dx have the dimension of length, $d\Omega(x)/dx$ is dimensionless. Since it is meaningful to say that a dimensionless quantity is much less than the dimensionless constant unity, (7.4) is meaningful.

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Problem for the Reader

What is the dimension of $\int p(x)dx$?

Solution

The wave number $p(x)$ has the dimension of the inverse of the distance. Since dx has the dimension of distance, the integral $\int p(x)dx$ is dimensionless. We may add that the integral $\int p(x)dx$ appears in the exponent of the WKB solutions, and an exponent should always be a dimensionless quantity.

We shall now explain what is the physical quantity we must compare the wavelength to. Let $p(x)$ in (7.1) be equal to $a^{-1}P(x/L)$, where both a and L have the dimension of length, and where $P(x/L)$ is dimensionless and is of the order of unity. Then $\Omega(x) = 2\pi a/P(x/L)$. This says that the magnitude of $\Omega(x)$ is of the order of $2\pi a$. Note that x/L is dimensionless and by $\Omega(x)$ being a function of x/L we imply that L is the scale characterizing the variation of the wavelength as a function of x. This means that the derivative of $\Omega(x)$ is of the order of $1/L$ times $\Omega(x)$. Therefore, $d\Omega(x)/dx$ is of the order of $2\pi a/L$. As a result, the inequality (7.4) is satisfied if

$$a/L \ll 1,$$

or

$$a \ll L.$$

This says that the WKB approximation is valid if the wavelength is small compared to $2\pi L$, where L is the length characterizing the scale of the variation of the wavelength.

As a trivial example, if p is a constant, the wavelength does not change no matter how much x varies. Thus L is equal to infinity and the wavelength is much smaller than L. Indeed, if p is a constant, the WKB solutions are not only good approximate solutions of (7.1), but the exact solutions of (7.1).

Let us revisit the case of (7.5), for which the WKB approximation has been justified. In this case, we have $a = \lambda^{-1}$and $L = 1$. Hence a is much less than L.

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Problem for the Reader

Consider (7.1) with $p(x) = P(\epsilon x)$, where $\epsilon \ll 1$ and $P(\epsilon x)$ is of the order of unity, i.e.,

$$\frac{d^2y}{dx^2} + P^2(\epsilon x)y = 0.$$

Note that the coefficient of y in this equation is not large, but varies slowly with x. Can we apply the WKB approximation to this equation?

Solution

The WKB approximation is valid if (7.4) is satisfied. Since

$$\frac{d}{dx}\frac{1}{P(\epsilon x)} = -\epsilon\frac{P'(\epsilon x)}{P^2(\epsilon x)},$$

(7.4) is satisfied if ϵ is very small and P does not vanish. Therefore, we conclude that we may apply the WKB approximation to (7.1) with $p(x) = P(\epsilon x)$ when x is not near a zero of $P(\epsilon x)$. We note that in this case, $a = 1$ and $L = \epsilon^{-1}$. Thus the wavelength is again much smaller than the characteristic length L.

The case of $p(x) = P(\epsilon x)$ and the case of $p(x) = \lambda P(x)$ are actually related by a change of the scale of the independent variable. To wit, let

$$X = \epsilon x;$$

then eq. (7.1) with $p(x) = P(\epsilon x)$ becomes

$$\frac{d^2y}{dX^2} + \lambda^2 P^2(X)y = 0,$$

where the large parameter λ is equal to ϵ^{-1}. This says that if we use X as the independent variable, p is in the form $\lambda P(X)$.

We mention a couple of physical problems in which the WKB approximation is useful. Consider the problem of determining the shadow cast on a wall by a point light source in front of a screen. To obtain the exact solution of this problem, one solves the wave equation and makes the solution satisfy the boundary conditions imposed by the presence of the screen. This is a difficult boundary-value problem. On the other hand, the shadow on the wall is very accurately determined simply by drawing straight lines from the light source to the edges of the screen. This is because when the wavelength of light is very small compared to the dimensions of the screen, the WKB approximation can be used to justify the results obtained with the use of geometric optics.[1] As another example, we know that Newtonian mechanics is an approximation of quantum mechanics. However, the behavior of atoms obeying the rules of wave mechanics is drastically different from that of particles obeying the rules of Newtonian mechanics. How does one reconcile these two sets of rules? The answer again lies in the WKB approximation, in which the Schrödinger equation is reduced to the Hamilton-Jacobi equation satisfied by the classical action of Newtonian mechanics.

The WKB approximation can also be used to solve problems in which the functional behavior is rapidly growing or rapidly decaying rather than rapidly oscillatory, an example being the problems of boundary layer, which we will discuss in Chapter 9. Consider the equation

$$y'' - \eta^2 y = 0. \tag{7.16}$$

The WKB solutions are given by

$$y^{\pm}_{WKB}(x) = \frac{1}{\sqrt{\eta(x)}} e^{\pm \int \eta(x) dx}. \tag{7.17}$$

These solutions are good approximations of the solutions of (7.16) if

$$\left| \frac{d}{dx} \frac{1}{\eta(x)} \right| \ll 1. \tag{7.18}$$

The counterpart of (7.5) is

$$\eta(x) = \lambda N(x), \tag{7.19}$$

where $\lambda \gg 1$ and $N(x)$ is of order unity. As before, if η is in the form (7.19), the inequality (7.18) is always satisfied unless x is near a zero of $N(x)$.

B. Solutions Near an Irregular Singular Point

In some mathematical problems, the large parameter λ is not explicitly exhibited. As an example, consider the problem of solving the equation $y'' + xy = 0$ when x

[1] S. I. Rubinow and T. T. Wu, *Journal of Applied Physics* 27:1032 (1956); T. T. Wu, *Physical Review* 104:1201 (1956).

is very large. In this problem, x inherently contains a large parameter. Indeed, let x be of the order of Λ, with $\Lambda \gg 1$. We may put

$$x \equiv \Lambda X,$$

where X is of the order of unity. Then the Airy equation is

$$\left(\frac{d^2}{dX^2} + \Lambda^3 X \right) y = 0.$$

This is in the canonical form for which the WKB method can be applied.

Thus the WKB approximation is useful for obtaining the asymptotic solutions near an irregular singular point of a second-order linear homogeneous equation. While we have already given a method in the preceding section to obtain these solutions, it applies only when the rank of the singular point is an integer. The WKB method has no such restriction. In addition, the use of the WKB method makes it easy to obtain the leading terms of the asymptotic series.

Let us consider the leading asymptotic terms for the solutions of the equation

$$y'' + xy = 0. \tag{7.20}$$

For $x < 0$, we have, comparing with (7.16),

$$\eta = (-x)^{1/2}.$$

Thus the WKB solutions are

$$|x|^{-1/4} e^{\pm 2|x|^{3/2}/3}. \tag{7.21}$$

We conclude immediately that when x is large and negative, one of these solutions is an exponentially increasing function of x and the other is an exponentially decreasing function of x.

When x is positive, we have, comparing with (7.1),

$$p = x^{1/2}.$$

Thus the WKB solutions are

$$x^{-1/4} e^{\pm i 2 x^{3/2}/3}, \tag{7.22}$$

both being oscillatory functions of x.

Next we will give the entire asymptotic series for when x is positive and very large in magnitude.

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Problem for the Reader

Find the asymptotic series for the solutions of the equation (7.20) for $x \gg 1$.

Solution

We note from the power function of the exponent in the WKB solutions that the rank of a singular point at infinity is $3/2$. Since the rank is not an integer, the method given in the preceding chapter cannot be directly applied. By (7.4), these WKB solutions are good approximate solutions if

$$x \gg (1/2)^{2/3}.$$

To find corrections of the WKB solutions, we put

$$y = \exp\left(\pm \frac{2x^{3/2}}{3} i\right) v.$$

Then v satisfies the equation

$$\left(\frac{d}{dx} + \frac{1}{4x}\right) v = \pm \frac{i}{2x^{1/2}} \frac{d^2}{dx^2} v. \tag{7.23}$$

The dimension of the operator on the left side of the equation above is -1, while the dimension of the operator on the right side is $-5/2$. These two dimensions differ by $3/2$, which is not an integer. Thus we make the change of variable

$$\rho \equiv x^{3/2}.$$

In terms of the variable ρ, these two dimensions differ by unity, an integer. Then (7.23) becomes

$$\left(\frac{d}{d\rho} + \frac{1}{6\rho}\right) v = \pm \frac{3i}{4} \left(\frac{d}{d\rho} + \frac{1}{3\rho}\right) \frac{dv}{d\rho}. \tag{7.24}$$

Let

$$v = \sum a_n \rho^{-n-s}, \ a_0 \neq 0, \ \text{ and } \ a_{-1} = a_{-2} = \cdots = 0. \tag{7.25}$$

We have

$$\frac{dv}{d\rho} + \frac{v}{6\rho} = \sum - \left(n + s - \frac{1}{6}\right) a_n \rho^{-n-s-1},$$

and

$$\pm\frac{3i}{4}\left(\frac{d}{d\rho}+\frac{1}{3\rho}\right)\frac{dv}{d\rho}=\pm\frac{3i}{4}\sum\left[(n+s)(n+s+1)-1/3(n+s)\right]a_n\rho^{-n-s-2}$$
$$=\pm\frac{3i}{4}\sum(n+s-1)(n+s-1/3)a_{n-1}\rho^{-n-s-1}.$$

Thus we get

$$-\left(n+s-\frac{1}{6}\right)a_n=\pm\frac{3i}{4}(n+s-1)(n+s-1/3)a_{n-1}. \tag{7.26}$$

Setting $n=0$ in the equation above, we get

$$s=1/6.$$

We set $s=1/6$ in (7.26) and get, for $n>0$,

$$a_n=\mp\frac{3i}{4}\frac{(n-5/6)(n-1/6)}{n}a_{n-1}=\left(\mp\frac{3i}{4}\right)^n\frac{\Gamma(n+1/6)\Gamma(n+5/6)}{\Gamma(1/6)\Gamma(5/6)n!}a_0. \tag{7.27}$$

Therefore, for $x\gg 1$, the two asymptotic solutions of (7.20) are

$$x^{-1/4}\exp\left(\pm\frac{2x^{3/2}}{3}i\right)\sum_{n=0}^{\infty}\left(\mp\frac{3i}{4}\right)^n\frac{\Gamma(n+1/6)\Gamma(n+5/6)}{n!}x^{-3n/2}. \tag{7.28}$$

Problem for the Reader

Find the WKB solutions for the Bessel equation

$$\left(\rho^2\frac{d^2}{d\rho^2}+\rho\frac{d}{d\rho}+\rho^2-p^2\right)Y(\rho)=0. \tag{7.29}$$

Solution

First we transform eq. (7.29) into the form of (7.1). This is done by putting

$$Y(\rho)=\rho^{-1/2}y(\rho),$$

getting

$$\left(\frac{d^2}{d\rho^2} + 1 - \frac{p^2 - 1/4}{\rho^2}\right) y = 0.$$

Identifying

$$p = \sqrt{1 - \frac{p^2 - 1/4}{\rho^2}},$$

we have, when ρ is large,

$$p \simeq 1.$$

Thus we have

$$y_{WKB}(\rho) = e^{\pm i\rho}$$

and

$$Y_{WKB}(\rho) = \rho^{-1/2} \exp(\pm i\rho). \tag{7.30}$$

This is in agreement with the leading terms of (6.63) and (6.64), found with a little more effort.

The WKB solutions sometimes even help us to obtain the exact solution of an equation.

Problem for the Reader

Solve in closed form

$$y'' + \left(x^2 + \frac{2}{9x^2}\right) y = 0. \tag{7.31}$$

Solution

The WKB solutions of (7.31), valid for $|x| \gg 1$, are easily found to be

$$y_{WKB}(x) = x^{-1/2} \exp\left(\pm \frac{x^2 i}{2}\right). \tag{7.32}$$

The exponent in (7.32) is equal to a constant times x^2; thus infinity is a singular point of the ODE of rank two. (We see once again that using the WKB method is an easy way to obtain the rank of a singular point.)

Let us look into the possibility of y being a Bessel function. Comparing the exponent of the solution of (7.30) with that of (7.32) suggests to us that

$$\rho = \frac{x^2}{2}. \tag{7.33}$$

With this identification of the independent variables, the exponential function of the solution of (7.30) is now equal to that of (7.32). Yet the factor multiplying the exponential function of (7.30) is $\rho^{-1/4}$, which differs from that of (7.32) by a factor of $\rho^{-1/4}$. Let

$$Y = \rho^{-1/4} y; \tag{7.34}$$

then the asymptotic forms of Y are exactly the same as those given by (7.30). Therefore, we make the change of the independent variable (7.33) and the change of the dependent variable (7.34) for equation (7.31). Then it is straightforward to show that eq. (7.31) becomes the Bessel equation with order p equal to $1/12$. Thus the general solution of (7.31) is

$$y(x) = x^{1/2}\left[aJ_{1/12}(x^2/2) + bJ_{-1/12}(x^2/2)\right], \tag{7.35}$$

where a and b are constants.

Problem for the Reader

Show that the parabolic function $D_\nu(x)$ satisfying the equation

$$y'' + (\nu + 1/2 - x^2/4)y = 0$$

is not directly related to the Bessel functions unless $\nu = -1/2$.

Solution

Let

$$\eta = \sqrt{x^2/4 - \nu - 1/2}.$$

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When x is large and positive, we have

$$\eta \simeq x/2 - (\nu + 1/2)x^{-1}.$$

Thus

$$\int \eta dx \simeq x^2/4 - (\nu + 1/2)\ln x.$$

Note that while a term in η that is equal to a constant times x^{-1} is small when x is large, it generates a term in $\int \eta dx$ that is equal to a constant times $\ln x$. Such a term is large when x is large, and cannot be ignored. The WKB solutions for this equation are

$$y^{+}_{WKB}(x) = x^{\nu} e^{-x^2/4}$$

and

$$y^{-}_{WKB}(x) = x^{-\nu-1} e^{x^2/4}.$$

The power functions for these two solutions are different unless $\nu = -1/2$, while those for the Bessel functions are the same. Thus, unless $\nu = -1/2$, the parabolic cylinder function $D_\nu(x)$ is not equal to a power function times a Bessel function $Z_p(\rho)$, where p is any number and ρ is any function of x.

Finally, we mention that while one may get the notion that (7.4) is likely to hold in the region of x where $p(x)$ is very large, this is not always the case. Consider

$$p(x) = 1/x, \tag{7.36}$$

which is very large when x is small. Yet

$$\frac{d}{dx}\frac{1}{p(x)} = 1,$$

which is not small when x is small.

C. Higher-Order WKB Approximation

We shall in this section find the higher-order terms of the WKB approximation. For this purpose let us return to eq. (7.11). Since this equation has a small parameter ϵ and is linear, it is straightforward to use it to derive successive corrections to the WKB approximations. We put

$$v = v_0 + \epsilon v_1 + \epsilon^2 v_2 + \cdots, \tag{7.37}$$

where v_n, $n = 0, 1 \cdots$, are independent of ϵ. The series in (7.37) is called a perturbation series, which is expected to be useful when ϵ is small. We substitute (7.37) into (7.11) and get

$$(v_0 + \epsilon v_1 + \epsilon^2 v_2 + \cdots)' + \frac{P'}{2P}(v_0 + \epsilon v_1 + \epsilon^2 v_2 + \cdots)$$
$$= \pm \frac{i\epsilon}{2P}(v_0 + \epsilon v_1 + \epsilon^2 v_2 + \cdots)''. \tag{7.38}$$

In the lowest-order approximation we set ϵ in (7.38) to zero and get

$$v_0{}' + \frac{P'}{2P} v_0 = 0.$$

This equation gives

$$v_0(x) = \frac{1}{\sqrt{P(x)}},$$

which is, aside from an immaterial constant multiple, (7.13).

Setting to zero the sum of terms in (7.38) that are proportional to ϵ, we get

$$v_1' + \frac{P'(x)}{2P(x)} v_1 = \pm \frac{i}{2P(x)} \left(\frac{1}{\sqrt{P(x)}} \right)''. \tag{7.39}$$

Solving this first-order linear equation, we find that

$$v_1(x) = \pm \frac{i}{2\sqrt{P(x)}} \int \frac{1}{\sqrt{P(t)}} \left(\frac{1}{\sqrt{P(t)}} \right)'' dt. \tag{7.40}$$

Now we are ready to give a justification of the WKB method, which is approximating the solution of (7.1) by truncating the series of (7.37). Strictly speaking, truncating a series is justified if we succeed in proving that the sum of terms neglected is much less than the sum of terms kept. But proving this is sometimes difficult to do. We shall be content with proving that the $(n+1)^{\text{th}}$ term in the series is much less than the n^{th} term if ϵ is sufficiently small. Thus we will accept that the WKB solutions (7.8) are good first-order approximations if

$$|\epsilon v_1| \ll |v_0|. \tag{7.41}$$

Since ϵ is small, (7.41) is satisfied provided that $v_1(x)/v_0(x)$ does not blow up, which is true unless $P(x)$ happens to vanish.

If $P(x)$ vanishes at x_0, the differential equation (7.1) is said to have a turning point at x_0. At a turning point of the differential equation, we may prove from (7.40) that the ratio v_1/v_0 blows up, and the WKB approximation fails.

How far away from the turning point must it be in order for the WKB approximation to work? If when x is near x_0, $P(x)$ goes to zero like $(x-x_0)^n$, then $v_1(x)$ blows up like

$$(x - x_0)^{-1-3n/2}, \tag{7.42}$$

while $v_0(x)$ blows up like $(x - x_0)^{-n/2}$. Thus (7.41) requires

$$|x - x_0| \gg \frac{1}{\lambda^{1/(1+n)}}. \tag{7.43}$$

Aside from a multiplicative constant, (7.43) is the same condition as (7.15).

We may find all higher-order terms of the solution from (7.38). This is done by gathering all the terms in (7.38) proportional to ϵ^m and setting the sum to zero. We get

$$v_m' + \frac{P'}{2P} v_m = \pm \frac{i}{2P} v_{m-1}''. \tag{7.44}$$

Thus

$$v_m(x) = \pm \frac{i}{2\sqrt{P(x)}} \int \frac{dx}{\sqrt{P(x)}} \frac{d^2}{dx^2} v_{m-1}(x). \tag{7.45}$$

From (7.45), we obtain the m^{th}-order term of the perturbation series once the $(m-1)^{\text{th}}$-order term of the perturbation series has been found. Thus we obtain all v_m by successive iteration.

If $P(x)$ has no zero, all v_m are finite. When ϵ is sufficiently small, we have

$$|\epsilon v_m| \ll |v_{m-1}|. \tag{7.46}$$

Thus the WKB approximation is justified to higher orders. Here we like to give the reader a reminder: The WKB approximation has been justified to higher orders only if $p(x)$ is of the form (7.5) or $\eta(x)$ is of the form of (7.19), and neither of them vanishes. (I feel obligated to say it as I have seen the WKB approximations being too liberally applied.)

We may show that if $P(x)$ vanishes at x_0 and is given by (7.14) near x_0, the condition (7.46) is satisfied provided that x is sufficiently far away from x_0 so that (7.15) is satisfied. (See homework problem 3.)

The high-order WKB approximations for (7.16) can be obtained in a similar way. We put

$$y \equiv e^{\pm \int \eta(x) dx} v. \tag{7.47}$$

Then we have

$$v' + \frac{\eta'}{2\eta} v = \mp \frac{1}{2\eta} v''. \tag{7.48}$$

We may use (7.48) to obtain successive approximations of the WKB solutions of (7.16).

D. Turning Points

As we have mentioned, the WKB approximation is useful in problems of wave propagation. In this section we demonstrate this by applying it to the wave equation that governs the quantum mechanical behavior of a particle.

Consider the time-independent Schrödinger equation of one spatial dimension discussed at the end of Chapter 5. We will write this equation in the form

$$\frac{d^2\phi}{dx^2} + \lambda^2 \left[E - V(x)\right] \phi = 0, \tag{7.49}$$

where λ is equal to $\sqrt{2m}$ divided by the Planck constant, with m the mass of the particle. We shall consider λ as a very large number.

As we have mentioned, $|\phi|^2$ is the probability density of the particle, E is the energy of the particle, and $V(x)$ is the potential. We note that $E - V$ is equal to the kinetic energy of the particle.

In the region where

$$E > V(x),$$

the kinetic energy of the particle is positive. Comparing with (7.1), we identify $p(x)$ with

$$\lambda\sqrt{E - V(x)}.$$

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The WKB solutions are

$$p^{-1/2}e^{i\int pdx} \text{ and } p^{-1/2}e^{-i\int pdx}. \tag{7.50}$$

Thus the wavefunctions ϕ in the classically accessible region are oscillatory as a function of x.

In the region where

$$E < V(x),$$

the kinetic energy of the particle is negative, and the momentum of the particle is imaginary. In classical mechanics, the momentum is not allowed to be imaginary. Thus this region is inaccessible to the classical particle. In quantum mechanics, the wavefunction ϕ is exponentially small in this region. The WKB solutions of the wave equation are

$$\eta^{-1/2}e^{\int \eta dx} \text{ and } \eta^{-1/2}e^{-\int \eta dx}, \tag{7.51}$$

where

$$\eta = \lambda\sqrt{V(x) - E}.$$

The first solution above is exponentially increasing as x increases, and the second solution above is exponentially decreasing as x increases. Since a particle is rarely observed in the classically inaccessible region, we require the wavefunction ϕ in this region to be the solution that decreases to smaller and smaller values as x goes deeper and deeper into the classically inaccessible region.

Let there be a point x_0 at which

$$E - V(x_0) = 0.$$

Note that x_0 is the point at which the momentum of the particle vanishes. By (7.4), the WKB approximation fails at x_0.

Let us study the behavior of the wavefunction near x_0. We shall assume that $V'(x_0)$ is different from zero; hence $E-V(x)$ is negative at one side of x_0 and positive at the other side of x_0. The point x_0 is the dividing point between a classically accessible region and a classically inaccessible region. In classical mechanics, the particle cannot move into the region where its kinetic energy is negative, and must turn back as it arrives at x_0. That is why x_0 is called a turning point of the wave equation. As we may expect, the qualitative behavior of the quantum wavefunction goes through a transition near x_0. More precisely, the WKB solution changes from an oscillatory behavior from one side of the turning point to an exponential behavior at the other side of the turning point. However, we cannot continue the solution from one side of the turning point to the other side of the turning point with the WKB solutions alone. This is because the WKB approximation fails in a small neighborhood around the turning point x_0.

Fortunately, another approximation for the solution is available when x is close to x_0. In order to make the following discussions appear as simple as possible, we shall choose $x_0 = 0$.

Near the turning point that is chosen to be the origin, we have by assumption

$$[E - V(x)] \approx -\alpha x,$$

with α equal to $V'(0)$. This linear approximation is valid for

$$|x| \ll 1. \tag{7.52}$$

We will call the region given by (7.52) the turning region. In the turning region, the wavefunction ϕ is approximately described by

$$\frac{d^2\phi}{dx^2} - \alpha\lambda^2 x\phi = 0, \tag{7.53}$$

the general solution of which is a linear superposition of Airy functions.

We expect the WKB solutions to be good when x is sufficiently far away from the turning point. How far away from the turning point should x be in order for the WKB approximation to be valid? Not very far, as it turns out. Indeed, by (7.4), the region where the WKB approximation is valid is determined to be

$$|x| \gg \Lambda^{-2/3}, \tag{7.54}$$

where

$$\Lambda^2 = |\alpha|\lambda^2.$$

Since Λ is very large compared to unity, $\Lambda^{-2/3}$ is very small compared to unity. Thus the WKB solutions are good even when x is close, although not too close, to the turning point.

This means that there are values of x for which both (7.52) and (7.54) are fulfilled. These are the values of x satisfying the inequality

$$1 \gg |x| \gg \Lambda^{-2/3}.$$

The regions of x satisfying the inequality above will be called the overlapping regions, in which both the WKB solutions and the Airy function solutions are good

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approximations of the wavefunction. Because x may be either positive or negative, there are two disjoint overlapping regions. They are

$$1 \gg x \gg \Lambda^{-2/3} \tag{7.55a}$$

and

$$1 \gg -x \gg \Lambda^{-2/3} \tag{7.55b}$$

Visually, these two overlapping regions form the two fringes of the turning region.

The existence of these two overlapping regions is crucial. The Airy functions approximate the wavefunction well throughout the turning region, which overlaps with the region in which the WKB solutions are good. This fact enables us to join the WKB solution from one fringe to the other, using the Airy functions to interpolate the wavefunction through the central part of the turning region where the WKB approximation fails.

We now demonstrate specifically how this is done. First we consider the case in which (7.49) has only one turning point. Without loss of generality we shall let α be positive. (For if α is negative, we may make a change of variable, referring to x as $-x$.) If α is positive, the kinetic energy of the particle is negative in the region $x > 0$. In classical mechanics, this is the region inaccessible to the particle. Thus we require the wavefunction ϕ to vanish rapidly as x increases in this region. The WKB solution in the region $x > 0$ satisfying this requirement is

$$\phi_{WKB}(x) = \frac{e^{-\int_0^x \eta(x')dx'}}{\sqrt{\eta(x)}}. \tag{7.56}$$

Incidentally, we may choose the solution to be a constant times the right side of (7.56), but this affects only the overall normalization of the wavefunction and we will leave it the way it is.

We use the solution of (7.56) to describe the wavefunction in the region $x > 0$. This WKB solution fails as x gets into the interior of the turning region. To obtain an approximation of the solution that is good throughout the turning region, we note that when x is in the turning region, we may approximate (7.49) by (7.53). By changing the independent variable, we transform eq. (7.53) into the Airy equation

$$\frac{d^2}{d\rho}y - \rho y = 0,$$

where

$$\rho = \Lambda^{2/3} x. \tag{7.57}$$

Thus the two independent solutions of (7.53) are the Airy functions denoted by $Ai(\rho)$ and $Bi(\rho)$.

Note that ρ is related to x by a change of scale. Therefore, a region of unit width in the x variable corresponds to a region of width $\Lambda^{2/3}$ in the ρ variable. Since $\Lambda^{2/3}$ is a very large number, the region of unit width in the x variable corresponds to a region of very large width in the ρ variable. When we study the solution as a function of ρ, it is like studying the function with a magnifier, under which the scale is enlarged.

In terms of the variable ρ, the turning region is given by

$$|\rho| \ll \Lambda^{2/3}, \tag{7.58}$$

which is a very large region, and the overlapping regions are given by

$$1 \ll \rho \ll \Lambda^{2/3} \tag{7.59a}$$

and

$$1 \ll -\rho \ll \Lambda^{2/3}. \tag{7.59b}$$

As we will discuss in the next chapter, the asymptotic forms of $Ai(\rho)$ when the magnitude of ρ is large are given by

$$Ai(\rho) \sim \frac{e^{-2\rho^{3/2}/3}}{2\sqrt{\pi}\rho^{1/4}}, \quad \rho \to \infty, \tag{7.60}$$

and

$$Ai(\rho) \sim \frac{\sin\left[\frac{2}{3}(-\rho)^{3/2} + \frac{\pi}{4}\right]}{\sqrt{\pi}(-\rho)^{1/4}}, \quad \rho \to -\infty. \tag{7.61}$$

Since $Ai(\rho)$ vanishes exponentially when ρ is large and positive, the Airy function solution that matches the WKB solution (7.56) is

$$\phi(x) = cAi(\rho), \tag{7.62}$$

where c is a constant.

To determine c, we match the solution (7.62) with the solution (7.56) in the first overlapping region (7.59a) or, equivalently, (7.55), where both approximations are good. Now when x is small and positive, we have

$$\eta(x) \approx \Lambda x^{1/2} = \Lambda^{2/3}\rho^{1/2}$$

and

$$-\int_0^x \eta(x')dx' \approx -\frac{2\rho^{3/2}}{3}.$$

Thus the solution (7.62) and the solution (7.56) have the same functional form in the overlapping region. They match exactly if we choose

$$c = \frac{2\sqrt{\pi}}{\Lambda^{1/3}}.$$

Therefore, as we continue the wavefunction (7.56) into the turning region, the wavefunction becomes

$$\phi(x) = \frac{2\sqrt{\pi}}{\Lambda^{1/3}} Ai(\rho). \tag{7.63}$$

Next we continue the wavefunction further into the region $x < 0$. By (7.61), the Airy function solution (7.63) as ρ is negative and large is

$$\phi(x) \approx \frac{2}{\Lambda^{1/3}} \frac{\sin\left[\frac{2}{3}(-\rho)^{3/2} + \frac{\pi}{4}\right]}{(-\rho)^{1/4}}. \tag{7.64}$$

In the second overlapping region (7.55b) or, equivalently, (7.60b), we have

$$p(x) \approx \Lambda(-x)^{1/2} = \Lambda^{2/3}(-\rho)^{1/2}$$

and

$$\int_x^0 p(x')dx' \approx \frac{2}{3}(-\rho)^{3/2}.$$

Therefore, the WKB solution for $x < 0$ that matches (7.64) in the second overlapping region is

$$\frac{2\sin\left[\int_x^0 p(x')dx' + \pi/4\right]}{\sqrt{p(x)}}. \tag{7.65}$$

We point out that as it is being continued into the region $x < 0$, the WKB solution (7.56) for $x > 0$ turns into neither a multiple of the WKB solution

$$p^{-1/2}e^{i\int p dx}$$

nor a multiple of the WKB solution

$$p^{-1/2}e^{-i\int p dx}.$$

Instead, it turns into a linear superposition of equal magnitude of these two solutions.

With (7.56), (7.63), and (7.65), we have, for the wave equation of one turning point, an approximate solution covering all values of x. It is easy to generalize the results above to the case in which the turning point is located at x_0 rather than at the origin. We simply call $X = x - x_0$, and the point $x = x_0$ corresponds to the point $X = 0$. Thus we only need to replace x in the results above by X. I remind the reader that if we re-express the results so obtained in terms of the variable x, the limits of integration given in the formulae above should be changed accordingly. For example, $X = 0$ corresponds to $x = x_0$.

We are now ready to treat the wave equation with two turning points. Let x_0 and x_1 be the two turning points, with $x_0 > x_1$. Let $E - V(x)$ be positive inside the region $x_1 < x < x_0$, and be negative outside the region. In classical physics, the region $x_1 < x < x_0$ is the region to which the particle is confined. Thus we must require the wavefunction to decrease rapidly as x leaves the region $x_1 < x < x_0$. In particular, we require that the wavefunction vanish as x approaches plus infinity or minus infinity.

This is a boundary-value problem with the trivial solution

$$\phi = 0.$$

This trivial solution satisfies both (7.49) and the conditions of vanishing at plus infinity and minus infinity.

A nontrivial solution exists only if E takes some special values called eigenvalues. These eigenvalues can be found approximately with the WKB method, as we shall presently show.

Since it is required to vanish as $x \to -\infty$, the WKB solution in the region $x < x_1$ is chosen to be

$$\frac{e^{-\int_x^{x_1} \eta(x')dx'}}{\sqrt{\eta(x)}}. \qquad (7.66)$$

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Since it is required to vanish as $x \to \infty$, the WKB solution in the region $x > x_0$ is of the form

$$\phi_{WKB}(x) = a\frac{e^{-\int_{x_0}^{x}\eta(x')dx'}}{\sqrt{\eta(x)}}, \tag{7.67}$$

where a is a constant. Since the normalization of the wavefunction has been set by choosing the wavefunction in the region $x < x_1$ to be precisely the expression in (7.66), we no longer have any freedom to choose a for the wavefunction in the region $x > x_0$. The constant a will be determined with a calculation.

We shall give the wavefunction only in the region that is inside the interval $x_0 > x > x_1$ but is sufficiently far away from the turning points x_0 and x_1. The WKB approximation holds throughout this region. Since the matching with the Airy functions has already been performed, we shall be spared the chores of doing it once again if we are not interested in the wavefunction near the turning points.

We find with the use of (7.65) that the WKB solution in the region $x_0 > x > x_1$ that matches with (7.66) is

$$\frac{2\sin\left[\int_{x_1}^{x} p(x')dx' + \pi/4\right]}{\sqrt{p(x)}}.$$

We also find with (7.65) that the WKB solution in the region $x_0 > x > x_1$ that matches with (7.67) is

$$\frac{2a\sin\left[\int_{x}^{x_0} p(x')dx' + \pi/4\right]}{\sqrt{p(x)}},$$

where the factor a is the same as in (7.67). Since these two solutions are valid in the same region $x_1 < x < x_0$, they are required to be the same. Hence we have

$$\sin\left[\int_{x_1}^{x} p(x')dx' + \pi/4\right] = a\sin\left[\int_{x}^{x_0} p(x')dx' + \pi/4\right]. \tag{7.68}$$

Setting $x = x_0$, we get from (7.68)

$$\sin(I + \pi/4) = a/\sqrt{2}, \tag{7.69}$$

where

$$I \equiv \int_{x_1}^{x_0} p(x')dx'.$$

Differentiating (7.68) with respect to x and setting $x = x_0$, we get

$$\cos(I + \pi/4) = -a/\sqrt{2}. \tag{7.70}$$

Taking the ratio of (7.69) and (7.70), we get

$$\tan(I + \pi/4) = -1.$$

Therefore,

$$I = \int_{x_1}^{x_0} p(x)dx = (n + 1/2)\pi, n = 0, 1, 2, \cdots, \tag{7.71}$$

which is known as the Bohr quantization rule giving the approximate energy eigenvalues. Substituting this value of I into (7.68), we get

$$a = (-1)^n.$$

Problem for the Reader

With the use of the WKB method, find the quantum energy eigenvalues of the harmonic oscillator for which

$$V(x) = \frac{1}{2}\kappa x^2,$$

where κ is a constant.

Solution

The turning points are obtained by setting

$$E - \frac{1}{2}\kappa x^2 = 0,$$

which gives

$$x_1 = -L,\ x_0 = L,$$

where

$$L = \sqrt{\frac{2E}{\kappa}}.$$

We have

$$I = \lambda \int_{-L}^{L} \sqrt{E - \kappa x^2/2}dx.$$

To evaluate the integral above, it is best to scale the variable of integration so that the limits of integration are -1 and 1. Thus we put

$$X = x/L,$$

and get

$$I = \lambda E \sqrt{\frac{2}{\kappa}} \int_{-1}^{1} \sqrt{1 - X^2} dX.$$

Since

$$\int_{-1}^{1} \sqrt{1 - X^2} dX = \pi/2,$$

we find

$$I = \lambda E \pi \sqrt{\frac{1}{2\kappa}}.$$

Therefore, Bohr's quantization rule says that

$$E_n = \sqrt{2\kappa}\lambda^{-1}(n + 1/2), n = 0, 1, 2 \cdots ,$$

which turns out to be the exact answer.

We note that in classical physics, the energy of a harmonic oscillator can take any value from zero to infinity. But in quantum mechanics, the energy can only take the discrete values given above. The lowest of the energy eigenvalues is

$$\frac{1}{2}\sqrt{2\kappa}/\lambda,$$

while the other energy eigenvalues are evenly spaced with distance $\sqrt{2\kappa}/\lambda$ apart. We say that the energy of a harmonic oscillator is quantized. (A more in-depth discussion of the solution near a turning point will be given in Section E of Chapter 9.)

Homework Problems for This Chapter

Solutions to the Homework Problems can be found at www.lubanpress.com.

1. Show that the Wronskian of y^+_{WKB} and y^-_{WKB} given by (7.4) is a constant.
Hint: To calculate the Wronskian of y^+_{WKB} and y^-_{WKB}, it is efficient to use the formula (1.7).

2. The WKB solutions (7.8) can also be derived by putting $y = e^{i\lambda S}$ and substituting this expression for y into

$$\frac{d^2y}{dx^2} + \lambda^2 P^2(x) y = 0.$$

a. Show that S satisfies the nonlinear second-order differential equation

$$i\lambda S'' - \lambda^2 (S')^2 + \lambda^2 P^2 = 0.$$

b. Explain why we may drop the term iS'' in the equation above and obtain

$$(S')^2 - P^2 = 0.$$

This equation is known as the Hamilton-Jacobi equation. Show that this approximation is justified as long as

$$|\lambda S''| \ll (\lambda S')^2.$$

c. Show that the solutions of the Hamilton-Jacobi equation are $S = \pm \int P dx$, which will yield the zeroth-order WKB approximation. Show also that, with S given by $\pm \int P dx$, the inequality given in (b) is the same as (7.7).

d. Obtain the additional factor $1/\sqrt{P(x)}$ in the WKB solutions by expressing S as the perturbation series

$$S = S_0 + \epsilon S_1 + \cdots$$

and solving for S_1, where $\epsilon = \lambda^{-1}$ is a small number.

e. Discuss how to obtain the second-order WKB approximation with this approach. Compare this method of getting the second-order WKB approximation with the method given in Section C of this Chapter.

3. Show that when $P(x)$ has a zero of order n at x_0, (7.46) is true if (7.15) is satisfied, where n can be any positive number. What if $P(x)$ vanishes at $x = 0$ like e^{-1/x^2}?

4. Find the zeroth-order and the first-order terms for the solutions of

$$\frac{d^4y}{dx^4} + \lambda^4 U(x) y = 0, \quad \lambda \gg 1,$$

where $U(x)$ does not vanish. Consider both the cases $U(x) > 0$ and $U(x) < 0$. How do you find the higher-order terms of the solutions?

5. Solve the following equations in closed forms:

a. $y'' + x^m y = 0,$
b. $y'' + (x^2 + 3x^{-2}/16)y = 0,$
c. $y'' - (x^4 - 3x^{-2}/16)y = 0.$

6. Obtain the WKB solutions for the equations below. For what positive values of t are these approximations good? Can you solve them in closed forms?

a. $\dfrac{d^2y}{dt^2} + (1 + e^{-\epsilon t})y = 0, \text{ where } \epsilon \ll 1.$
b. $\dfrac{d^2y}{dt^2} + e^{-\epsilon t}y = 0, \text{ where } \epsilon \ll 1.$
Hint: To obtain the solutions of these equations in closed forms, put

$$\tau = \frac{2}{\epsilon}e^{-\epsilon t/2}.$$

The solutions for the equation of (a) are $J_{\pm 2i/\epsilon}(\tau)$, whereas those of (b) are $J_0(\tau)$ and $Y_0(\tau)$.

7. For the example in (6b), make the transformation $T = \epsilon t$ and cast the equation into the form

$$\frac{d^2y}{dT^2} + \lambda^2 P^2(T)y = 0.$$

What is λ and what is $P(T)$? Show that $P(T)$ has only one zero. What is the order of this zero of $P(T)$?

8. Consider $y'' + x^{-2}y = 0$ for x of the order of ϵ, $\epsilon \ll 1$. Let $x \equiv \epsilon X$ and show that there is no large parameter in the resulting equation. What do you conclude from this example?

9. Find the WKB solutions of the confluent hypergeometric equation

$$xy'' + (c - x)y' - ay = 0,$$

where a and c are constants. Determine the values of x for which the WKB solutions are good approximations.

10. With the use of the WKB method, find the approximate quantum energy eigenvalues of a particle moving in the potential

$$V(x) = \frac{1}{2}\kappa x^4,$$

where κ is a constant.

Hint: The integral

$$\int_0^1 \sqrt{1 - X^4} dX$$

can be expressed by a Beta function

$$B(p, q) = \int_0^1 t^{p-1}(1-t)^{q-1} dt \quad (p, q > 0),$$

which is equal to

$$\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$$

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